October 31, 2017 LTH 1140, NSF-ITP-17-135, MCTP-17-21

Renormalizable, asymptotically free gravity without ghosts or tachyons

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We analyse scale invariant quadratic quantum gravity incorporating non-minimal coupling to a multiplet of scalar fields in a gauge theory, with particular emphasis on the consequences for its interpretation resulting from a transformation from the Jordan frame to the Einstein frame. The result is the natural emergence of a de Sitter space solution which, depending the gauge theory and region of parameter space chosen, can be free of ghosts and tachyons, and completely asymptotically free. In the case of an SO(10) model, we present a detailed account of the spontaneous symmetry breaking, and we calculate the leading (two-loop) contribution to the dilaton mass.

I. INTRODUCTION

There has been increasing interest in the past few years in finding alternatives to the common lore concerning the fundamental interactions. With no sign of supersymmetric particle production (as yet) at the LHC, the idea that weak-scale susy might be the solution to the hierarchy problem of the Standard Model (SM) has become less attractive. Secondly, especially since it seems that our universe may well have a positive cosmological constant, the relationship of string theory to cosmology seems ever more remote. The landscape of string theory vacua has difficulty accommodating de Sitter-like backgrounds. Further, since the asymptotic behavior of such spacetimes is not flat, there is no S-matrix. Motivated by these observations, in a series of recent papers [1–5], we have explored the properties of a renormalizable [6], asymptotically-free [7], classically scale-invariant, quantum field theory (QFT) of gravity, including matter fields in such a way that *all* couplings remain asymptotically free (AF). Asymptotic freedom allows one to entertain the possibility that this is an ultraviolet (UV) completion of gravity and that there is no new physics to be discovered at higher scales. It also allows one to make perturbative, controllable calculations at arbitrarily high energy scales[1]. Even though the QFTs we study are not truly scale invariant because of the conformal anomaly, it is attractive to assume that the models are classically scale invariant since such theories are technically natural [11] in the sense that it is not necessary to fine-tune power-law divergent loop corrections in order to stabilize their scalar mass spectra. Under these circumstances, all masses, including the Planck mass M_P and the cosmological constant Λ, arise via dimensional transmutation (DT) [12]. (Such a program was already proposed in Ref. [7].)

Contrary to the widespread belief that renormalizable gravity violates unitarity, having both a spin-two ghost as well as a spin-zero tachyon in flat background, we claim that, in a de Sitter (dS) background, these models have no unstable fluctuations for a certain range of couplings. (This was already known for the theory without matter [13]. See Sec. III.). There remain five zero modes which, we have argued [5], correspond to collective modes that are unphysical and, similar to gauge modes, do not contribute to on-shell observables. Thus, although these zero modes are a generic feature of all such models in a dS background, they are not a barrier to stability. Our assertion is limited to quadratic order in the fluctuations, the same order at which claims of instabilities and ghosts have been made. We do not know whether, in higher order when interactions are included, this will remain true. This is closely related to the question of unitarity, since we do not have a canonical action or a Hamiltonian that guarantee unitary evolution.

In previous work [1–4], we have displayed models exhibiting DT for a range of couplings, within which there is a subset of values such that the extrema are local minima of the Euclidean action. We have also satisfied the constraints on the couplings so that the Euclidean path integral (EPI) is convergent for all values of the fields. We found that these minima lie within the basis of attraction of the AF fixed point gauge model with a "Higgs" field in the adjoint representation, for a certain fermion content [4]. So far, we have only described the spectrum of this model qualitatively. In this paper, we wish to discuss the physics of this model near or below the scale of symmetry breaking. In the process, we shall also substantiate our claim that the fluctuations are stable. For this purpose, as is often the case, it will prove useful to pass from the Jordan frame to the Einstein frame.

To set the stage and review our conventions, we begin

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[1] Such models have been termed "Totally Asymptotically Free" in Ref. [8]. The notion of AF is distinct from nonperturbative "asymptotic safety" [9], which has undergone a resurgence in recent years; see *e.g.,* Ref. [10].

with the action for gravity without matter. The action for renormalizable gravity can be written in several different equivalent forms; we take it (in the Jordan frame) as

$$S_{ho}^{(J)} = \int d^4x \sqrt{g_J} \left[\frac{C^2}{2a} + \frac{R^2}{3b} + c\, G \right], \tag{1.1}$$

where $C_{\kappa\lambda\mu\nu}$ is the Weyl tensor, R is the Ricci scalar, and G is the so-called Gauss-Bonnet (G-B) term, $G \equiv C^2 - 2W$, where $W \equiv R_{\mu\nu}^2 - R^2/3$. We shall work with the Euclidean form of the metric with the convention for the Ricci tensor $R_{\mu\nu}$ in which $R > 0$ corresponds to positive curvature[2]. To this must be added a matter action, which will be discussed in due course. Euclidean dS space is the S^4 sphere. This may be regarded as a submanifold of flat, Euclidean space in five-dimensions. From this perspective, the radius of the S^4 sphere is $r_0 = \sqrt{12/R_0}$, where R_0 is the value of the Ricci scalar on-shell.

The metric $g_{\mu\nu}$ of Eq. (1.1), in transverse-traceless (TT) gauges, includes five on-shell tensor modes as well as a scalar mode (dilaton), plus an additional four modes that are gauge-dependent. So this may be thought of as a scalar-tensor theory of gravity. One may add an Einstein-Hilbert (E-H) term $-M_P^2 R/2$ as well as a cosmological constant $M_P^2 \Lambda$. Since they are UV-irrelevant, their presence does not affect renormalizability or AF, although issues of fine-tuning may re-emerge, at least in non-supersymmetric models. This is what has usually been done in the past, but, in the scenarios that we have described in which scalar matter is added in a classically scale-invariant fashion, such terms are not needed so long as DT occurs at a scale where the dimensionless couplings are sufficiently small that perturbative calculations remain reliable.

Over the years, there have been numerous papers involving higher-derivative gravity in a similar spirit to ours, some of which attempt to provide a complete QFT of gravity [7, 8, 14, 15], possibly conformal and/or supersymmetric [16, 17], while others attempt to generate the Planck mass dynamically along the lines of induced gravity [18, 20–22]. This subject has been reviewed in Ref. [17, 19]. These references are just a sample, and, given the extensive literature about higher-derivative gravity, spanning more than 50 years, we shall have to limit further citations to those that are of immediate relevance.

An outline of the subsequent sections is as follows: In the next section, we discuss aspects of the stability of the model in de Sitter background, the controversy over the sign of b, and some of the difficulties establishing that QFTs of this sort are (or are not) unitary. In Sec. III, we review the addition of matter in the Jordan frame, taking up the simplest example of the real scalar field, while in Sec. IV, we transform the same model to the Einstein frame in order to elaborate on several points not discussed in our previous papers. Then in Sec. V, we apply these methods to the case of the $SO(10)$-model, which is a prototype for any such non-Abelian gauge theory coupled to gravity. In Sec. VI, we turn to the issue of spontaneous symmetry breaking (SSB) in this model, emphasizing the differences from a similar calculation in the Jordan frame. Then we embark upon a discussion of the resulting particle spectrum in this model for the vector bosons (Sec. VII), the heavy scalars (Sec. VIII), the curvature fluctuations (Sec. IX) and finally the dilaton mass (Sec. X) arising at two-loop order. Following some remarks on the resulting low-energy effective field theory (Sec. XI), we summarize our results and discuss open questions in Sec. XII. There follow two appendices with details useful in the body of the text. In Appendix A, we review how the curvature tensor transforms under conformal transformations, and in Appendix B our Lie algebra notation and the form of the model after SSB to $SU(5)\otimes U(1)$.

II. STABILITY, ASYMPTOTIC FREEDOM, AND UNITARITY

Everyone who has considered renormalizable gravity agrees that $a > 0$ is necessary and sufficient for this coupling to be AF. As we have previously mentioned [4], the appropriate sign of b has been subject to some dispute, and we shall take up this issue below.

We adopt the assumptions of Euclidean quantum gravity [23] to the extent that they are known. To some extent, these have been reviewed in Ref. [17, 19]. Our philosophy is very close to that elaborated by Christensen and Duff [24] and by Avramidi [13][3]. A basic tenet of this approach is that the Euclidean path integral (EPI) be convergent for all values of the fields. Unlike E-H gravity, integrating over conformal modes presents no special difficulties. This requires both a and b in Eq. (1.1) to be positive for sufficient large scales where the "classical" approximation is valid. This appears to be a minimal requirement for the existence of candidates for stable "vacuum" states in QFT. In flat spacetime, the requirement that the Euclidean action be bounded below together with certain others [25], eventually allows for analytic continuation to Lorentzian signature with an action that respects CPT invariance and unitarity. Whether something similar is true for the extension of gravity given in Eq. (1.1) is not known. It should not be difficult to extend reflection-positivity to Euclidean renormalizable gravity, but cluster decomposition obviously must

[2] Because the variation of G vanishes, the term C^2 can be replaced by $2W$ in Eq. (1.1). This often simplifies some tensor algebra.

[3] These will be further reviewed below. Ref. [24] does not consider renormalizable gravity, and Ref. [13] mentions the inclusion of matter only in passing.

be modified for a compact spacetime such as S^4. For Euclidean spacetimes without boundaries, this would imply that there are not degenerate no-particle states. In particular, apparently degenerate no-particle states must have finite tunnelling amplitudes between them so that they can be superposed. For example, this is familiar in flat space when there are finite action solutions of the classical equations of motion (EoM) for Euclidean signature (instantons). In that case, there are degenerate no-particle states in perturbation theory for which, as a result of non-perturbative affects, the degeneracy is removed.

There exist persistent doubts about unitarity in this class of theories. Unitarity is certainly suspect in theories with actions containing both quadratic curvature terms of the kind exhibited in Eq. (1.1) *and* an explicit linear term $-M_P^2 R$, because of the following observations, which were raised originally in Ref. [6]. In the presence of a non-zero Planck mass M_P, the propagator in flat space contains a term in the tensor mode that behaves as

$$\frac{1}{q^2(q^2+M_P^2)} = \frac{1}{M_P^2}\left(\frac{1}{q^2} - \frac{1}{q^2+M_P^2}\right). \qquad (2.1)$$

Thus, if the graviton term $1/q^2$ has the usual sign, the second term corresponds to a massive, spin two particle with negative kinetic energy, i.e., a ghost. Further, in the scalar sector, there remains a particle with mass [6] $m_0 = \sqrt{-b}\,M_P/2$, where $M_P = 1/\sqrt{8\pi G_N}$ is the so-called "reduced" Planck mass or string scale. Thus, there is a tachyon instability for $b > 0$ in flat background, a primary reason some have argued that $b < 0$. Yet, as remarked above, $b > 0$ is the sign required for convergence of the EPI. Since, however, we have demonstrated (and will confirm here) that the phase of our model having terms both linear and quadratic in curvature exists only below a definite scale that is determined by DT, the argument based on Eq. (2.1) does not apply. We shall make some further comments about unitarity below.

Another argument suggesting that $b < 0$ would be preferable goes as follows: One adds to R^2 a term with an auxiliary field χ

$$\frac{1}{3b}R^2 \mp \frac{1}{2}\left(\chi^2 - \frac{\xi}{2}R\right)^2 = \left[\frac{1}{3b} \mp \frac{\xi^2}{8}\right]R^2 \pm \frac{\xi\chi^2}{2}R \mp \frac{\chi^4}{2}. \qquad (2.2)$$

with ξ an arbitrary "coupling constant." The sign of ξ must be chosen to be the same as the sign of $\langle R\rangle$, so that $\langle\chi\rangle^2 = \xi\langle R\rangle/2$ has a solution for real $\langle\chi\rangle$. The sign of the added term must be chosen to be opposite to the sign of b, so that the coefficient of R^2 on the RHS can be taken to vanish ($\xi^2 = 8/|3b|$). Thus, it seems that the original term in the Lagrangian density proportional to R^2 is equivalent to a non-minimal gravitational coupling of a scalar field together with its self-interaction. We then see that if $b < 0$, the linear term in R corresponds to attractive gravity, and the "potential term" χ^4 is bounded below. This is frequently used [26] to argue that the sign demanded physically is $b < 0$. This sign is the opposite of that required for convergence of the EPI and for AF of b.

We *do not*, however, subscribe to this popular belief that $b(\mu) < 0$ (for sufficiently large scales μ) because the field χ, unlike an independent dynamical degree of freedom (DoF), is inextricably linked to the scalar curvature, *i.e.,* $\chi^2 = \xi R/2$. From the point of view of the EPI, the preceding construction is misleading; one cannot simply add such a term and integrate over χ since, having insisted $b(\mu) > 0$ at large scale, the integral over χ would diverge. To introduce an auxiliary field, one must actually add to the integrand of the EPI a term proportional to $\mathcal{D}\chi^2\,\delta(\chi^2 - \xi R/2)$, or its equivalent.

To confirm the fallacy in such arguments, consider the far simpler situation in ordinary ϕ^4 field theory in flat spacetime with potential $V(\phi) = m^2\phi^2/2 + \lambda\phi^4/4$. It is generally believed that, in order to have a sensible ground state, one must have the renormalized coupling $\lambda(\mu) > 0$, at least for some range of relatively large scales[4]. Following a procedure similar to the previous one, we write

$$\frac{\lambda}{4}\phi^4 \mp \frac{1}{4}\left(\sigma - \xi\phi^2\right)^2 = \frac{1}{4}\left(\lambda \mp \xi^2\right)\phi^4 \pm \frac{\xi\sigma}{2}\phi^2 \mp \frac{1}{4}\sigma^2. \qquad (2.3)$$

σ is an auxiliary field[5] for which $\langle\sigma\rangle = \xi\langle\phi\rangle^2$. To be able to cancel the ϕ^4 term on the RHS, thereby reducing the action for ϕ from quartic to quadratic, we must choose the sign of the added term to be opposite to that of λ. For the "potential term" σ^2 to be bounded below, the last term must be positive. By the logic above, we ought then to demand $\lambda < 0$, the very opposite of what we required initially!

We conclude that one may not treat an auxiliary field such as σ as if it can be taken "off-shell" for fixed values of the other fields on which it depends. Conversely, it may also not be consistent to discuss the behavior of a dynamical field such as ϕ for arbitrary values of the auxiliary field. The construction is also wrong in detail, because the equation $\xi(\mu)^2 = \lambda(\mu)$ is not in fact correct for arbitrary μ; in short, it is not renormalization group invariant[6] (RGI). Similarly, in the gravitational case, the relation $3b(\mu) = 8/\xi(\mu)^2$ is not RGI. In sum, although one may introduce an auxiliary field in the manner outlined here, one can be misled drawing conclusions based on treating it as an independent DoF.

As an aside, this same issue arises in other models involving polynomials in R of even higher degree, so-called $f(R)$ models of gravity. It seems that a similar sign error afflicts many of those treatments in the literature[7].

[4] The sign of λ is a renormalization group invariant since $\lambda = 0$ yields free field theory. $\lambda > 0$ is IR-free and not AF, so this must be regarded as an effective field theory.

[5] Note that, with this definition, σ has dimensions of mass-squared.

[6] For further discussion on this point, see, e.g., Sec. II of Ref. [27].

[7] For reviews of such models, see e.g., Refs. [26]. For further extensions of this method, see Ref. [28]. More recently, Narain [29] has argued that there is a conflict between the Lorentzian and Euclidean formulations. Our expectation would be that, once again, this is a reflection of a similar sign issue.

In this paper, we shall assume that the cosmological constant of the effective field theory at low energy is positive, so we shall only be concerned with de Sitter-like solutions of the model. That assumption happens to be correct in the classically scale-invariant theories we have studied, although we have not investigated whether it must be true in all such theories.

An effort similar to ours embracing a classically scale-invariant action for both matter and gravity has been called "Agravity" [20]. However, our approach is fundamentally different inasmuch as these authors insist that $b < 0$ for the reasons reviewed above. Given that the ($b < 0$) model is no longer AF, they treated renormalizable gravity as an effective field theory. It is an improvement over the E-H theory in the same way that the electroweak theory is an improvement over the Fermi model and may allow some speculations about physics beyond the Planck scale. More recently [21], by considering non-perturbative possibilities rather than adding new dynamical degrees of freedom, they have speculated that perhaps the non-AF theory is correct to infinite energy. We prefer to explore the possibility that the AF model ($b > 0$) is the completion of the E-H theory, that perturbation theory continues to hold, and no new physics is required at higher scales, which, we contend, would be a far more compelling result.

As we have indicated in past work [4] and has been emphasized long ago in Refs. [24] and [14], further complications and opportunities arise in the presence of a cosmological constant, even though the curvature may be small. In that case, flat space is not a solution to the EoM, so some of the foregoing issues may disappear. Our point of view overlaps with that adopted by Avramidi [13], who explicitly included M_P and Λ in his action and who emphasized that, so long as his couplings and masses obeyed certain inequalities, neither the scalar nor the tensor modes present instabilities. In the present notation, he showed that the tensor modes are stable and ghost-free for $a > 0$, $\Lambda > 0$, and $2/(3b) < 1/a + M_P^2/(16\Lambda)$. Moreover, there is no instability in the scalar sector provided $M_P^2/(16\Lambda) < 2/(3b)$, which is compatible with the tensor constraint. These inequalities can even be satisfied in the classically scale-invariant case where $M_P \to 0$ (for fixed Λ.) When matter is included[8], for the cases we studied [1, 3, 4], the inequalities were modified, but there still existed regions of parameter space where there were no instabilities.

Nevertheless, in calculating the one-loop correction to dS space, there remain five zero modes that seem to be universally present in both Einstein gravity and in renormalizable gravity, with or without the inclusion of matter. As we have reviewed elsewhere [5], these so-called non-isometric, conformal Killing modes have a rather long history. We have argued that these reflect a collective mode that, in four-dimensions, is peculiar to the S^4 manifold. If so, they will be present not just at one loop but to all orders in perturbation theory. However, unlike other occurrences of such coherent motions, we claim the corresponding collective degrees of freedom (DoF), i.e., the "center of mass" coordinates, are unphysical and not relevant to the determination of the stability of dS background. They nevertheless do enter into the calculation of various gauge-invariant quantities, such as the on-shell effective action. The essential issue is whether or not there is a collective coordinate missing from the effective action. We presume not.

As a result of the foregoing, we believe that there exists a renormalizable, theory of gravity that, when matter fields are included, can yield new models that (1) undergo DT in perturbation theory, (2) yield a positive cosmological constant, (3) are locally stable for a range of couplings, and (4) are AF in all couplings. So far, we have confirmed this for only one such model [4], but it is surely not unique. The issue of unitarity remains unresolved, but it is far more subtle than has been treated thus far in the present context. For example, one of the lessons from considering QFT in curved spacetime [30, 31] is that the so-called no-particle state can appear completely different to observers in different frames, resulting in the definition of particle states correspondingly different.

In the next section, we expand on the way in which these results have been achieved. In previous papers, we used the renormalization group to determine the one-loop effective action. This method makes some assumptions that direct calculations via path integrals avoid. In the next section, we shall discuss this in the simplest case, that of a real scalar field [3], but most of these points apply to the non-Abelian case as well, as will be discussed in Sec. V.

III. INCLUDING MATTER FIELDS IN THE JORDAN FRAME

We discussed DT in pure gravity, Eq. (1.1), in Ref. [1], and shall not repeat that here. Matter can be added in many forms, and our goal is to focus on non-Abelian models, in particular, on the $SO(10)$ model discussed in Ref. [4]. However, there are a few points that can be more easily stated in the simplest case, that of the real scalar field [3]. It is also easier to have the experience of transforming to the Einstein frame in that case, as we shall do in Sec. IV, before proceeding to the non-Abelian gauge theory in Sec. V. For pedagogical reasons, then, we shall first reconsider the real field, taking the opportunity to clarify certain points omitted from our earlier paper.

To the action in Eq. (1.1), we add the action for a

[8] For $b < 0$, the one-loop corrections to the effective action in dS background should have an imaginary part, reflecting an instability. This is another reason that we believe that Agravity [20] is not self-consistent.

single, real field ϕ :

$$S_{cl}^{(J)} \equiv S_{ho}^{(J)} + S_{m}^{(J)}, \tag{3.1a}$$

$$S_{m}^{(J)} \equiv \int d^4x\sqrt{g_J}\left[\frac{1}{2}(\nabla\phi)^2 + \frac{\lambda}{4}\phi^4 - \frac{\xi\phi^2}{2}R\right]. \tag{3.1b}$$

Defining the rescaled couplings $y \equiv \lambda/a$, $x \equiv b/a$, we showed in Ref. [3] that this model in dS background has a single ultra-violet fixed point (UVFP) at $\xi = 0, y = 0, x \approx 39.8$. Given that all couplings are AF, the classical approximation ought to be increasingly accurate the higher the scale. In that paper, we derived the form of the one-loop corrections to the effective action using RGI, the known β-functions and generic form of the corrections in dS background. However, this "short-cut" has its limitations. It does not necessarily reveal all constraints on the couplings and would not produce the imaginary part present if the perturbative corrections were unstable. These can only be revealed by explicitly calculating the one-loop effective action. Here, we shall review that calculation via the EPI in a "classical" background field given by $\widehat{g}_{\mu\nu}(x), \varphi(x)$. For our purposes, it will suffice to consider the corrections on mass shell, i.e., where the effective action has extrema. To zeroth order, i.e., classically, the on-shell values of neither φ nor R can be known since the classical action $S_{cl}^{(J)}$ is scale invariant; however, the dimensionless ratio ϕ^2/R can be fixed. The first-variation of the classical action gives

$$\frac{\delta S_{cl}^{(J)}}{\delta\phi} = -\Box\phi + \lambda\phi^3 - \xi\phi R, \tag{3.2}$$

$$\begin{aligned}-\frac{\delta S_{cl}^{(J)}}{\delta g^{\mu\nu}} = \frac{1}{6a}\Big[&4RR_{\mu\nu} - 12R^{\kappa\lambda}R_{\mu\kappa\nu\lambda} + \\ &g_{\mu\nu}\big(3R_{\kappa\lambda}^2 - R^2\big) + \big(2\nabla_\mu\nabla_\nu R + g_{\mu\nu}\Box R - 6\Box R_{\mu\nu}\big)\Big] + \\ &\frac{2}{3b}\left[\frac{g_{\mu\nu}}{4}R^2 - RR_{\mu\nu} + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)R\right] - \frac{1}{2}\Big[T_{\mu\nu} - \\ &\xi\phi^2\Big[R_{\mu\nu} - g_{\mu\nu}R\Big] + \Big[\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\Big]\frac{\xi\phi^2}{2}\Big],\end{aligned} \tag{3.3}$$

where $T_{\mu\nu} \equiv \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}(\nabla\phi)^2 + \frac{\lambda}{4}\phi^4\right]$.

It is difficult to characterize the most general solution of these equations. Most sufficiently symmetric solutions of Einstein's equations continue to hold for these modified equations, such as the Schwarzschild and Schwarzschild-de Sitter solutions [32]. We can get a hint of what may be necessary if we take the trace of Eq. (3.3):

$$-2g^{\mu\nu}\frac{\delta S_{cl}^{(J)}}{\delta g^{\mu\nu}} = -\frac{4}{b}\Box R + (\nabla\phi)^2 + \lambda\phi^4 + \xi(3\Box - R)\phi^2. \tag{3.4}$$

(The terms in $1/a$ cancel out of the trace because of classical conformal invariance of the Weyl action.) Writing $\Box\phi^2 = 2\phi\Box\phi + 2(\nabla\phi)^2$, the right-hand side becomes

$$-\frac{4}{b}\Box R + (6\xi + 1)(\nabla\phi)^2 + 6\xi\phi\Box\phi + \lambda\phi^4 - \xi\phi^2 R. \tag{3.5}$$

Only for the conformal values $b \to \infty, \xi = -1/6$ does Eq. (3.5) become identical to ϕ times Eq. (3.2). On the other hand, there are solutions other than the conformal limit that are mutually compatible with the vanishing of both Eqs. (3.2), (3.3). For example, in the case that ϕ and R are constant (corresponding to Euclidean dS space,) both equations are satisfied when $\lambda\phi^2 = \xi R$.

Our first goal here is to make more explicit the requirements for calculating the one-loop effective action. Using the standard background field method of quantization by the path integral[9], we expand the classical action $S_{cl}^{(J)}$, Eq. (3.1), about a generic background by writing $\phi(x) = \varphi(x) + \delta\phi(x)$, $g_{\mu\nu} \equiv \widehat{g}_{\mu\nu}(x) + h_{\mu\nu}(x)$, expanding in a Taylor series about $\varphi(x), \widehat{g}_{\mu\nu}(x)$, and dropping the term linear in the "quantum fields" $\delta\phi(x), h_{\mu\nu}(x)$. The one-loop correction is obtained from the terms second order in the fluctuations $\delta\phi(x), h_{\mu\nu}(x)$. The tensor $h_{\mu\nu}$ can be conveniently decomposed in the transverse traceless (TT) gauge

$$h_{\mu\nu} = h_{\mu\nu}^{\perp} + \widehat{g}_{\mu\nu}h/4 + \ldots, \tag{3.6}$$

where $h \equiv \widehat{g}^{\mu\nu}h_{\mu\nu}$ and $h_{\mu\nu}^{\perp}$ is transverse ($\widehat{\nabla}^\mu h_{\mu\nu}^{\perp} = 0$) and traceless ($\widehat{g}^{\mu\nu}h_{\mu\nu}^{\perp} = 0$). The other terms represented by the ellipses involve gauge-dependent vector and scalar modes. After a lengthy calculation, this procedure yields

$$\begin{aligned}S^{(2)} = \widehat{S}^{(0)} + \frac{1}{2}\int d^4x\sqrt{\widehat{g}}\Bigg[&\big(\widehat{\nabla}\delta\phi\big)^2 + \Big(3\lambda\varphi^2 - \xi\widehat{R}\Big)(\delta\phi)^2 \\ &-\delta\phi\frac{3\xi\varphi}{2}\left[\Delta_0\left[\frac{\widehat{R}}{3} - \frac{2\lambda\varphi^2}{3\xi}\right]\right]h + 2\xi\varphi\,\delta\phi\widehat{R}_{\mu\nu}h^{\perp\mu\nu} + \\ &\frac{3}{8b}h\left[\Delta_0\left[-\frac{\widehat{R}}{3}\right]\Delta_0\left[-\frac{b\xi\varphi^2}{4}\right]\right]h + \\ &\frac{1}{2a}h_{\mu\nu}^{\perp}\left[\Delta_2\left[\frac{a\xi\varphi^2}{2} + \frac{\widehat{R}}{3}\Big(1 - \frac{2a}{b}\Big)\right]\Delta_2\left[\frac{\widehat{R}}{6}\right]\right]h^{\perp\mu\nu} + \\ &CT + other\Bigg],\end{aligned} \tag{3.7}$$

where the background metric $\widehat{g}_{\mu\nu}$ is to be used for contractions and covariant derivatives. The terms represented by CT indicate implicit counterterms necessary to render the effective action finite after integration over the quantum fields. Those represented by *other* are gauge-dependent and vanish on-shell, i.e., when the background fields satisfy their EoM. The symbols $\Delta_j[X] \equiv -\Box_j + X$

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[9] This has been summarized in the present context in an appendix in Ref. [1].

for integer j involve the so-called constrained Laplacian, $\Box_j$, upon which we elaborate further below.

This expression is to be inserted into the EPI and the integral over the quantum fields $\delta\phi, h, h^{\perp}_{\mu\nu}$ performed. For a generic background, this cannot be done analytically, but, analogous to the flat space effective potential, for $\varphi = \varphi_0$ and $\widehat{R}_{\mu\nu} = R_0\,\widehat{g}_{\mu\nu}/4$ with φ_0, R_0 constant, the integral can be carried out. If we further require that the background be on-shell, $\lambda\varphi_0^2 = \xi R_0$, the quadratic action for the fluctuations can be put into the form

$$\begin{aligned}\delta^{(2)} S^{(2)}_{os} = \frac{1}{2}\int d^4x\sqrt{\widehat{g}}\Bigg[&\delta\phi\Delta_0\Big[2\xi R_0\Big]\delta\phi - \\ &\delta\phi\frac{3\xi}{2}\sqrt{\frac{\xi R_0}{\lambda}}\,\Delta_0\Big[-\frac{R_0}{3}\Big]h + \\ &\frac{3}{8b}h\left[\Delta_0\Big[-\frac{R_0}{3}\Big]\Delta_0\Big[-\frac{b\xi^2 R_0}{4\lambda}\Big]\right]h + \\ \frac{1}{2a}h^{\perp}_{\mu\nu}\bigg[\Delta_2\Big[\frac{a\xi^2 R_0}{2\lambda}+\frac{R_0}{3}\Big(1-\frac{2a}{b}\Big)\Big]&\Delta_2\Big[\frac{R_0}{6}\Big]\bigg]h^{\perp\,\mu\nu}\Bigg],\end{aligned} \tag{3.8}$$

where the CT and $other$ terms have been suppressed. We take the background to be the sphere S^4 with curvature R_0. In flat five-dimensional Euclidean space, this corresponds the four-sphere of radius $\mathrm{r}_0 \equiv \sqrt{12/R_0}$ and angular volume $\omega_4 = 8\pi^2/3$. Thus the Euclidean spacetime volume $V \equiv \omega_4 \mathrm{r}_0^{\,4} = 384\pi^2/R_0^2$, is finite in this approximation. Following Ref. [7, 13], we expand in normalized eigenfunctions of the "constrained" Laplacian $\Box_j$ in order to determine whether the modes are stable and to be able to deal with the mixing between $\delta\phi$ and h. $\Box_j = \widehat{g}^{\mu\nu}\widehat{\nabla}_\mu\widehat{\nabla}_\nu$, where $\widehat{\nabla}_\mu$ represents the covariant derivative acting on a field of "spin" j. For example, $\Box_0$ represents the Laplacian on the background manifold acting on a scalar field such as $\delta\phi$. $\Box_1$ represents the Laplacian acting on a conserved vector field, ε^μ with $\widehat{\nabla}_\mu\varepsilon^\mu = 0$. $\Box_2$ represents the Laplacian on the background vector bundle acting on tensor fields such as $h^{\perp}_{\mu\nu}$, which is transverse and traceless. (Further details with references to the literature can be found in Ref. [7], summarized in Ref. [13].) Since the S^4 sphere is compact, the eigenvalues of the elliptic operator $-\Box_j$ are discrete and nonnegative. Explicitly, they are given by

$$\begin{aligned}-\Box_j Y^{nj}_{\ell,m} &= \mathrm{r}_0^{-2}\lambda_{nj}Y^{nj}_{\ell,m}, \\ \lambda_{nj} &= n(n+3)-j, \quad n=j,j+1,\ldots.\end{aligned} \tag{3.9}$$

for $n, j \geq 0$. The indices (ℓ, m) denote the various states of the degenerate eigenvalue. We shall not need their precise definitions; we just need to know the total degree of degeneracy [33], $d_{nj} = (2n+3)(2j+1)\big((n+1)(n+2) - j(j+1)\big)/6$, $n\geq j\geq 0$. For a scalar field $j=0$, $\lambda_{n0} = n(n+3)$ is simply the value of the quadratic Casimir of the angular momentum generators of $SO(5)$. It has degeneracy $d_{n0} = (2n+3)(n+1)(n+2)/6$.

Expanding the fluctuations in terms of the eigenfunctions Y^{nj}, normalized to one on the unit S^4,

$$\begin{aligned}\frac{1}{\omega_4}\delta^{(2)}S_{os} = \frac{3\xi}{ay}\sum_{n=0} d_{n0}\Bigg[&2\Big[\lambda_{n0}+24\xi\Big]\left(\frac{\delta\phi_n}{\varphi_0}\right)^2 - \\ &3\xi\Big[\lambda_{n0}-4\Big]\frac{\delta\phi_n}{\varphi_0}h_n + \\ &\frac{y}{16x\xi}\Big[\lambda_{n0}-4\Big]\left[\lambda_{n0}-\frac{3x\xi^2}{y}\right]h_n^2\Bigg] + \\ \frac{1}{8a}\sum_{n=2} d_{n2}\left[\lambda_{n2}+\frac{6\xi^2}{y}+4\left(1-\frac{2}{x}\right)\right]&\Big[\lambda_{n2}+2\Big]h^{\perp\,2}_n,\end{aligned} \tag{3.10}$$

where $\omega_4 \equiv 8\pi^2/3$, $y \equiv \lambda/a$. It is the ratios y and x that approach finite UVFPs. As in pure gravity, we take $a > 0$ so that it will be AF. Because the λ_{nj} monotonically increase with n, the modes will certainly be nonnegative for n sufficiently large but finite. Hence, we just need to determine whether a finite number of modes are stable.

First, however, we must deal with the fact that each of these sums formally diverge as $n \to \infty$ and are rendered finite by adding renormalization counterterms that have not been explicitly included above[10]. Regardless of how renormalization is carried out, instabilities at low n for arbitrarily large scale μ will not be removed. They introduce singularities for certain values of the couplings, which renormalization does not do and which, because of AF, will not be removed in higher order. Zero modes, such as those associated with $\lambda_{10} = 4$, must be subtracted and dealt with separately and will be discussed below.

Let us begin our stability analysis with the tensor modes $h^{\perp\,2}_n$ in Eq. (3.10). The lowest mode has n=2 for which $\lambda_{22} = 8$. Hence, the factor $\lambda_{n2}+2$ will be positive for all n, but the first factor will be positive only for $2+\xi^2/y > 4/(3x)$. Given the aforementioned properties of the UVFP, together with the information that ξ^2/y actually vanishes as the UVFP is approached, this inequality is easily satisfied at sufficiently high scales, so all the tensor modes are stable, at least at sufficiently high scale. An instability at lower scales would be associated with a phase transition.

What about the scalar ($j = 0$) modes? We must determine under what conditions the quadratic form in $(\delta\phi_n, h_n)$ is nonnegative. For $n = 0$, $\lambda_{00} = 0$, so this is simply

$$\frac{9\xi^2}{ay}\left[16\left(\frac{\delta\phi_0}{\varphi_0}\right)^2 + 4\frac{\delta\phi_0}{\varphi_0}h_0 + \frac{1}{4}h_0^2\right] \tag{3.11}$$

This quadratic form has one eigenvalue equal to $+585\xi^2/(4ay)$, which is positive since[11] $y > 0$. Its eigenvector has $(\delta\phi_0/\varphi_0, h_0) \propto (8, 1)$. The other eigenvalue is 0

[10] Given the forms of λ_{nj} and d_{nj} above, the zeta-function method [34] naturally comes to mind. This involves certain subtleties in applications such as this involving products of quadratic, elliptic operators, as reviewed in Ref. [35], but these should not affect our arguments.

[11] $y > 0$ is required for convergence of the EPI.

with eigenvector $(\delta\phi_0/\varphi_0, h_0) \propto (-1, 8)$. This zero mode is the dilaton and should have been anticipated: Under the assumption that the background field has nonzero curvature R_0 and nonzero scalar field φ_0, the classical scale invariance is spontaneously broken, so there must be a Goldstone boson. We can regard the preceding calculation as a purely classical determination of the eigenvalues for small fluctuations, so it must reflect this Goldstone mode. When we insert this into the EPI and integrate over the fields, this becomes the one-loop correction. In so doing, the zero mode must be factored out in order to obtain a finite result. To this order, this corresponds to a flat direction of the effective potential.

Since the scale invariance is anomalous and not a symmetry of the QFT, this mode can get a nonzero mass m_d in higher-order. Indeed, at two-loop order, we argued in Ref. [3] that $m_d^2 \neq 0$ and can be positive for some range of values of x, ξ, y. (See Eq. (10.15) below.) In particular, it is positive near the UVFP, so this zero mode ultimately does not destroy local stability.

The next mode is $n = 1$, for which $\lambda_{10} = 4$ with degeneracy 5. Clearly, the quadratic form degenerates to

$$\frac{24\xi}{ay}(6\xi+1)(\delta\phi_1/\varphi_0)^2. \tag{3.12}$$

Since $y > 0$ was required for stability of the $n = 0$ mode, we must therefore have $\xi > 0$ for stability of this mode. Obviously, its eigenvector $(\delta\phi_1/\varphi_0, h_1) \propto (1, 0)$. The second eigenvalue is obviously zero due to fluctuations in the direction $(0, 1)$. Thus, there are 5 zero modes associated with the fluctuation h_1 with $\delta\phi_1 = 0$. These existed already in the pure gravity case and are present in all models with S^4 background on-shell. As mentioned earlier, we have argued in Ref. [5] that these five zero modes are artifacts of the $SO(5)$ isometry of dS corresponding to an unphysical coherent fluctuation, a would-be collective mode corresponding to the motion of the center-of-mass coordinate of the S^4 sphere, so we expect these zero modes to persist to all orders in perturbation theory. They are not Killing vectors, but are conformal Killing vectors not usually associated with physical isometries of the action. They are peculiar to an S^4 background and even occur for the E-H action [24]. We have argued that these unique modes do not reflect an actual physically-allowed fluctuation. As they only exist for an S^4 background, it seems likely that more realistic models will not have such unphysical collective coordinates. Further research is required to determine whether some non- perturbative effect, such as tunneling to a background with a different topology, leads to a different background that removes such modes.

What about the $n = 2$ mode, for which $\lambda_{20} = 10$? The quadratic form becomes

$$\frac{6\xi}{ay}\left[2(5+12\xi)\left(\frac{\delta\phi_2}{\varphi_0}\right)^2 - 9\xi\frac{\delta\phi_2}{\varphi_0}h_2 + \frac{3y}{16x\xi}\left[10 - \frac{3x\xi^2}{y}\right]h_2^2\right]. \tag{3.13}$$

Both eigenvalues will be positive provided $\xi > 0$ and

$$\frac{y}{x} > \frac{3\xi^2(1+6\xi)}{2(5+12\xi)}. \tag{3.14}$$

Since $y > 0$ is required for convergence of the EPI, this inequality will be satisfied sufficiently near to the UVFP, i.e., to first order in ξ, y. All $n > 2$ eigenvalues are also positive. There is no guarantee that this continues to hold when nonlinear effects become important, e.g., if the UVFP were approached along a trajectory in violation of Eq. (3.14).

In sum, there are no unstable modes associated with the fluctuations, provided these inequalities are satisfied, as they are near the UVFP. At one loop order, there are 6 zero modes (or flat directions.) One is the scalar dilaton, which we shall show gets mass at two loops. The other five are associated with a coherent fluctuation that, we believe, should be regarded as unphysical.

Since we found no unstable modes, there will be no imaginary part to the one-loop correction. The result for the renormalized effective action is therefore the one given in our earlier paper [3] for this model, obtained by the renormalization group method.

The preceding remarks do not imply that the one-loop correction to the effective action cannot be negative at lower scales. In fact, our investigation [3] of the possibility of DT showed that it can indeed become negative. Unfortunately, we found the range of couplings for which the extremum was actually a minimum did not lie within the basin of attraction of the UVFP, so this model does not produce a physically useful result. This was one of several reasons that we proceeded to consider non-Abelian gauge theories, which are potentially more physically relevant anyway.

IV. TRANSFORMATION FROM THE JORDAN TO THE EINSTEIN FRAME

Most discussions of classical General Relativity proceed from the E-H action with minimal coupling. An action with non-minimal coupling, like the one discussed in the previous section, may under certain circumstances be transformed into a minimal coupling form by means of a conformal transformation of the metric. This is often referred to as going from the Jordan frame to the Einstein frame. Since this only involves a field redefinition, one might think that it is a simply matter of convenience, since interpretations of observables generally start from the Einstein frame. In the present context at least, we wish to argue that such a transformation is NOT so straightforward.

Given the classical action, Eq. (3.1), the conformal transformation is

$$\widetilde{g}_{\mu\nu} \equiv \Omega^{-2} g_{\mu\nu}, \quad \text{where} \quad \Omega^2 \equiv \phi^2/M^2, \tag{4.1}$$

and M is any convenient choice for the unit of mass. In any theory (and in the real world), the only observables are dimensionless ratios, so the choice for M is arbitrary[12] but fixed (i.e., not scale dependent). Such a transformation is permissible provided Ω neither vanishes nor is singular. In classically scale-invariant models, this is not at all trivial. In the path integral, the integration over $\phi(x)$ is formally over all real values at every point, so it is impossible to guarantee this in general unless one assumes that it is a set of measure zero. This can be argued in the context of the perturbation expansion in which $\phi(x) = \langle\varphi(x)\rangle+\delta\phi(x)$, assuming that the background field $\langle\varphi(x)\rangle$ is nowhere vanishing and that the perturbation $\delta\phi(x)/\langle\varphi(x)\rangle$ is in some sense small, so that it makes sense to assume $\varphi(x)\neq 0$ everywhere. Should the result of the calculation be that the on-shell background field vanishes anywhere, this construction would have to be revisited.

Assuming that $\phi(x) \neq 0$, the effects of the field redefinition in Eq. (4.1) on the various quantities in Eq. (3.1) are complicated. In Appendix A, we summarize the resulting changes on the various quantities entering the action. Defining $\xi' \equiv \xi+1/6$, $\zeta \equiv \sqrt{6\xi'}M\log(\phi/M)$, and $\vartheta_\mu \equiv \partial_\mu \log\phi = 1/(\sqrt{6\xi'}M)\partial_\mu\zeta$, we find[13]

$$S^{(E)}=\int d^4x\sqrt{\tilde{g}}\left[\frac{\lambda M^4}{4}-\frac{\xi M^2}{2}\widetilde{R}+\frac{1}{2}\big(\widetilde{\nabla}\zeta\big)^2+\mathcal{L}_{ho}\right], \quad (4.2a)$$

$$\text{where } \mathcal{L}_{ho}\equiv \frac{1}{2a}\widetilde{C}^2+\frac{1}{3b}\big(\widetilde{R}+6\widetilde{\nabla}\cdot\vartheta-6\vartheta_\mu^2\big)^2+c\,\widetilde{G}. \quad (4.2b)$$

On the one hand, we have simply performed a field redefinition, so one might expect the physics to be unchanged. On the other hand, the supposition that $\phi(x) \neq 0$ corresponds to SSB of scale invariance, so in fact, the physics is manifested quite differently in this broken phase. First of all, ζ plays the role of the dilaton, the (classical) Goldstone boson, which we previously identified in the Jordan frame from the mode expansion. (See discussion below Eq. (3.11).) As expected, ζ is derivatively coupled classically, so $\langle\zeta\rangle$, if constant, is arbitrary. (We shall find a convenient choice below in Eq. (10.9).) In principle, in the QFT, it may or may not be the case that $\langle\phi(x)\rangle \neq 0$. In fact, the issue of spontaneous breaking of scale invariance is actually moot in the QFT, because this is an anomalous symmetry. As a result, as mentioned earlier, this scalar will get a mass at two loops owing to the conformal anomaly.

The appearance of the dilaton field is just one consequence of the supposition that $\phi(x) \neq 0$. The matter action, Eq. (4.2a), takes the form of an E-H term linear in $\widetilde{R}$, with Planck mass-squared $M_P^2 \equiv \xi M^2$, plus a cosmological constant term with $M_P^2\Lambda \equiv \lambda M^4/4$, plus a term corresponding to the kinetic energy of the dilaton ζ.

The gravitational action, Eq. (4.2b), involves, in addition to the quadratic curvature terms, involves terms in various powers of $\nabla\zeta/M$. This is clearly extremely complicated, but it proves convenient to choose M to be on the order of the SSB scale v, where the one-loop correction has its minimum determined by DT. (See Ref. [3].) So long as $\xi(v)$ is in the range 0.1−10, this is also on order of the Planck mass, $M_P = \sqrt{\xi}M$. For small dilaton momenta, more precisely, when $\sqrt{\xi}\,\widetilde{\Box}\zeta \ll \sqrt{1+6\xi}\,M_P\widetilde{R}$, these terms may be neglected in first approximation. Then the entire dependence on the dilaton field is given by the matter action, Eq. (4.2a).

Although we have shown that DT can occur in this model, the values of the coupling constants required for this to occur with local stability of the associated scale does not lie within the basin of attraction of the UVFP in this model [3]. Consequently, we shall defer the determination by DT and the calculation of the dilaton mass to the $SO(10)$ model in the next section.

V. NON-ABELIAN GAUGE FIELD

We want to transform our $SO(10)$ model with a single adjoint scalar Φ [4] from the Jordan frame to the Einstein frame. Renormalizability will not be affected by a field redefinition and, for present purposes, a nonlinear transformation is useful. To review, our Jordan frame matter action is

$$\begin{aligned}S_m^{(J)} = \int d^4x\sqrt{g}\,\Big[&\frac{1}{4}\mathrm{Tr}[F_{\mu\nu}^2] + \frac{1}{2}\mathrm{Tr}[(D_\mu\Phi)^2] \\ &-\frac{\xi\mathrm{Tr}[\Phi^2]}{2}R + V_J(\Phi)\Big],\end{aligned} \quad (5.1)$$

The adjoint scalar field Φ is a 10×10 Hermitian matrix that may be decomposed as $\Phi = \sqrt{2}\,\phi_a R^a$, where the $\{\phi_a\}$ are real, and $\{R^a\}$ represents the 45 Hermitian generators of the fundamental or defining representation **10** of $SO(10)$. Similarly, the real, adjoint gauge field can be represented by $A_\mu = \sqrt{2}\,A_\mu^a R^a$, with the associated field strength $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]/\sqrt{2}$. The covariant derivative of Φ is $D_\mu\Phi \equiv \partial_\mu\Phi - ig[A_\mu, \Phi]/\sqrt{2}$. A brief review of our algebraic conventions is given in Appendix B.

In order to transform to the Einstein frame, we want to presume that the model undergoes SSB $\langle\Phi\rangle \neq 0$. The exact nature of the breaking will be worked out in subsequent sections. A nonlinear field redefinition will enable us to proceed in much the same way as in the case of the real singlet in the preceding section. We define $T_2 \equiv \mathrm{Tr}[\Phi^2] = \sum_a \phi_a^2$, and define

$$\Phi \equiv \Omega\,\Sigma \text{ with } \Omega^2 \equiv T_2/M^2, \quad (5.2)$$

in terms of an arbitrary unit of mass M. Then,

$$\mathrm{Tr}[\Sigma^2] = M^2. \quad (5.3)$$

Note that both T_2 and Ω are formally $SO(10)$ invariant.

[12] In theories having other mass parameters, such as the Planck mass M_P or scalar masses, M is usually chosen to some combination of those parameters. We leave it unspecified for now.

[13] We have dropped a surface term associated with $\nabla^2\zeta$.

One consequence of these definitions is that $\langle\Phi\rangle = \langle\Omega\rangle\langle\Sigma\rangle$, so that $\langle\Phi\rangle \neq 0$ if and only if both $\langle\Omega\rangle \neq 0$ and $\langle\Sigma\rangle \neq 0$. Although one may entertain other possibilities for SSB, they do not seem to be relevant in perturbation theory. Then

$$D_\mu\Phi = \Sigma\,\partial_\mu\Omega + \Omega\,D_\mu\Sigma \tag{5.4a}$$
$$\mathrm{Tr}[(D_\mu\Phi)^2] = M^2(\partial_\mu\Omega)^2 + \Omega^2\mathrm{Tr}[(D_\mu\Sigma)^2]. \tag{5.4b}$$

In passing from the first to the second line in Eq. (5.4), the cross term vanishes because

$$\begin{aligned}\mathrm{Tr}[\Sigma D_\mu\Sigma] = \mathrm{Tr}[\Sigma\partial_\mu\Sigma] - ig\mathrm{Tr}\big[\Sigma[A_\mu,\Sigma]\big] =& \\ \partial_\mu\mathrm{Tr}[\Sigma^2]/2 = \partial_\mu M^2/2 = 0,&\end{aligned} \tag{5.5}$$

where the term involving the gauge field A_μ vanishes by the cyclic property of the trace. The Jordan frame Lagrangian density, Eq. (5.1), then becomes

$$\begin{aligned}\mathcal{L}_m^{(J)} = \sqrt{g}\bigg[&\frac{1}{4}\mathrm{Tr}[F_{\mu\nu}^2] - \frac{\xi M^2\Omega^2}{2}R + \\ &\frac{M^2}{2}(\partial_\mu\Omega)^2 + \frac{\Omega^2}{2}\mathrm{Tr}[(D_\mu\Sigma)^2 + V_J(\Omega\,\Sigma)]\bigg],\end{aligned} \tag{5.6}$$

subject to the constraint $\mathrm{Tr}[\Sigma^2] = M^2$, Eq. (5.3).

The original field Φ provided a linear representation of a real adjoint multiplet and represented 45 DoF in the matter action Eq. (5.1). Evidently, in Eq. (5.6), one degree of freedom has been apportioned to Ω and only 44 DoF remain in Σ. This can be seen from Eq. (5.5), which implied that $\mathrm{Tr}[\Sigma\partial_\mu\Sigma] = 0$. Thus, the dynamical degrees of freedom associated with $\partial_\mu\Sigma$ are restricted to those "orthogonal" to Σ.

To complete this rewriting of the action Eq. (5.1), consider the potential, $V_J(\Phi)$. Defining $T_4 \equiv \mathrm{Tr}[\Phi^4] = \Omega^4\mathrm{Tr}[\Sigma^4]$, the potential is

$$V_J(\Phi) \equiv \frac{h_1}{24}T_2^{\,2} + \frac{h_2}{96}T_4 = \frac{h_1 M^4}{24}\Omega^4 + \frac{h_2}{96}\Omega^4\mathrm{Tr}[\Sigma^4]. \tag{5.7}$$

Thus, the only dependence of V_J on Σ is through T_4.

Further, the nonminimal coupling to the curvature in Eq. (5.6) is independent of Σ. As a result, the $SO(10)$ singlet Ω plays the role of the real scalar in the preceding section. Evidently, to transform to the Einstein frame, we need only suppose that $\langle\Omega\rangle \neq 0$ and can postpone the question of $\langle\Sigma\rangle$ until later. Without loss of generality (WLOG), we take $\langle\Omega\rangle > 0$. Then we can perform a conformal transformation, $\widetilde{g}_{\mu\nu} \equiv \Omega^{-2}g_{\mu\nu}$, to get the action in the Einstein frame

$$S_{ho}^{(E)} = \int d^4x\sqrt{\widetilde{g}}\left[\frac{\widetilde{C}^2}{2a} + \frac{1}{3b}\big(\widetilde{R} + 6\widetilde{\nabla}\cdot\vartheta - 6\vartheta_\mu^2\big)^2 + c\,\widetilde{G}\right], \tag{5.8a}$$

$$\begin{aligned}S_m^{(E)} = \int d^4x\sqrt{\widetilde{g}}\,\bigg[&\frac{1}{4}\mathrm{Tr}[\widetilde{F}_{\mu\nu}^2] - \frac{\xi}{2}M^2\widetilde{R} + \frac{(\widetilde{\nabla}\zeta)^2}{2} + \\ &\frac{h_1}{24}M^4 + \frac{1}{2}\mathrm{Tr}[(\widetilde{D}_\mu\Sigma)^2] + \frac{h_2}{96}\mathrm{Tr}[\Sigma^4]\bigg],\end{aligned} \tag{5.8b}$$

where, similar to the previous case, Eq. (4.2),

$$\zeta \equiv M\sqrt{6\xi'}\log\Omega, \quad \vartheta_\mu \equiv \widetilde{\partial}_\mu\log\Omega = \frac{1}{\sqrt{6\xi'}M}\widetilde{\partial}_\mu\zeta. \tag{5.9}$$

We must also keep in mind the constraint, Eq. (5.3).

One can show that the G-B term changes by a total divergence, $G \to \widetilde{G} + \nabla_\mu J^\mu$. (See Appendix A.) Although not the simplest form to quantize, the spectrum may be read off rather easily. The last line of Eq. (5.8b) shows that Σ is described by a gauged nonlinear sigma model with a scale-invariant self-interaction strength proportional to h_2. Regardless of the pattern of SSB, $\mathrm{Tr}[\Sigma^4] \geq (\mathrm{Tr}[\Sigma^2])^2/10 = M^4/10$, and it proves useful to rewrite the terms involving h_1, h_2 as the sum of two nonnegative terms

$$\frac{h_3 M^4}{24} + \frac{h_2}{96}\left[\mathrm{Tr}[\Sigma^4] - \frac{M^4}{10}\right], \tag{5.10}$$

where we defined $h_3 \equiv h_1 + h_2/40$.

Note that the action $S^{(E)} = S_{ho}^{(E)} + S_m^{(E)}$ is still formally invariant under the $SO(10)$ local gauge symmetry, since the conformal transformation employed only the gauge singlet $\Omega(x)$, which was presumed to have some non-zero vacuum expectation value (VEV) $\langle\Omega(x)\rangle$ to be determined. Although in principle, this can vary with position x^μ, there is a tacit assumption that $\Omega(x)$ vanishes nowhere since, otherwise, the transformed metric would degenerate somewhere. For simplicity, we shall seek SSB solutions in which $\langle\Omega\rangle \neq 0$, independent of x.

The role of the couplings h_1, h_2, (or h_2, h_3) in the Einstein frame suggests a dramatically different physical picture than that in the Jordan frame. In Eq. (5.8b), we can identify the Planck mass

$$M_P = \sqrt{\xi}\,M. \tag{5.11}$$

As in the case of the real field, we must have $\xi(v) > 0$ at the scale v of symmetry breaking in order for gravity to be attractive.

From Eq. (5.10), the "vacuum energy density" is $h_3 M^4/24$ or possibly larger, depending on the direction of SSB $\langle\Sigma\rangle$. In more conventional terms, the cosmological constant Λ corresponding to a vacuum energy density equal to $h_3 M^4/24$ is

$$\Lambda \equiv \frac{h_3}{24\xi^2}M_P^2. \tag{5.12}$$

Thus, we must have $h_3(v) > 0$ at the scale of symmetry-breaking in order for Λ to be positive.

The field ζ is the dilaton, which is massless in this approximation but will gain mass at two-loop order, $O(\hbar^2)$. (We shall determine its mass below in Sec. X.)

This is as much as can be said at the classical level about the singlets in Eq. (5.8). Further interpretation requires knowing more precisely the pattern of the breaking of $SO(10)$, which we shall discuss next.

VI. SPONTANEOUS SYMMETRY BREAKING OF SO(10)

In our parameterization, the direction of $SO(10)$ breaking is embodied in $\langle\Sigma\rangle$. Since the Σ field enters the Einstein frame action only via Eq. (5.8b), we can determine the possible extrema ignoring Eq. (5.8a), which is to say that they are essentially independent of the scale of SSB. In fact, we already showed in Ref. [4] that the only extremum that is a local minimum corresponds to breaking to $SU(5)\otimes U(1)$. In passing to the Einstein frame, we only utilized the singlet field $\Omega(x)$, so we would not expect this pattern to change. Indeed, unless there exists a sensible phase in which $\langle\Phi\rangle = 0$, the Einstein frame action, Eq. (5.8), must be completely equivalent to the Jordan frame action, $S^{(J)} \equiv S^{(J)}_{ho} + S^{(J)}_{m}$, Eqs. (1.1), (5.1). Although we could proceed by assuming this pattern of SSB is correct, it is illuminating to rederive it in the Einstein frame to confirm this expectation and to take note of the substantial differences from the Jordan frame.

For our purposes, it is convenient to make a unitary transformation to a basis in which the generators of $SO(10)$ take the form

$$R^a \equiv \frac{1}{\sqrt{2}}\left(\begin{array}{c|c} \mathcal{R}^a_1 & \mathcal{R}^a_2 \\ \hline \mathcal{R}^{a\,\dagger}_2 & -\mathcal{R}^{a\,\tau}_1 \end{array}\right) = \frac{1}{\sqrt{2}}\left(\begin{array}{c|c} \mathcal{R}^a_1 & \mathcal{R}^a_2 \\ \hline -\mathcal{R}^{a\,*}_2 & -\mathcal{R}^{a\,*}_1 \end{array}\right), \quad (6.1)$$

where[14] the $\mathcal{R}^a_j$ are 5×5 (complex) matrices with the properties that $\mathcal{R}^a_1$ is Hermitian, and $\mathcal{R}^a_2$ is antisymmetric. Hence, $\mathcal{R}^{a\,*}_1 = \mathcal{R}^{a\,\tau}_1$, where τ denotes the transpose. In this basis, unlike the original one, the Cartan subalgebra of $SO(10)$ can be diagonalized.

Correspondingly, we define for real components σ_a

$$\Sigma \equiv \sqrt{2}\,\sigma_a R^a = \left(\begin{array}{c|c} \sigma_a\mathcal{R}^a_1 & \sigma_a\mathcal{R}^a_2 \\ \hline \sigma_a\mathcal{R}^{a\,\dagger}_2 & -\sigma_a\mathcal{R}^{a\,*}_1 \end{array}\right) \quad (6.2a)$$

$$\equiv \left(\begin{array}{c|c} \Sigma_1 & \Sigma_2 \\ \hline {\Sigma_2}^{\dagger} & -{\Sigma_1}^{*} \end{array}\right). \quad (6.2b)$$

The constraint Eq. (5.3) implies

$$\sum_1^{45} \sigma_a^2 = M^2, \quad \text{or } \mathrm{Tr}[\Sigma_1^2 + \Sigma_2^{\dagger}\Sigma_2] = M^2/2. \quad (6.3)$$

Assuming $\langle\Sigma\rangle \neq 0$, one may utilize the $SO(10)$ symmetry of the action Eq. (5.8) to bring it to diagonal form

$$\langle\Sigma\rangle = \left(\begin{array}{c|c} \langle\Sigma_1\rangle & 0 \\ \hline 0 & -\langle\Sigma_1\rangle \end{array}\right), \quad (6.4)$$

where $\langle\Sigma_1\rangle$ is the matrix, Diag$\{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5\}$. These ς_i are the eigenvalues of $\langle\Sigma\rangle$, which are of course independent of the choice of basis. However, the basis chosen above, Eq. (6.1), is particularly convenient. Let us call the generators of the $SO(10)$ Cartan subalgebra H^i, with the corresponding generators of $SU(5)\otimes U(1)$ $\mathcal{H}^i$. Then we conclude that

$$\varsigma_i H^i = \langle\sigma_a\rangle R^a, \text{ and } \varsigma_i \mathcal{H}^i_1 = \langle\sigma_a\rangle \mathcal{R}^a_1. \quad (6.5)$$

With reference to the action Eq. (5.8b) and the constraint Eq. (6.7), in order to seek the extrema of the action, we must consider

$$\frac{h_2}{48}\mathrm{Tr}[\langle\Sigma_1\rangle^4] - \frac{\eta}{2}\mathrm{Tr}[\langle\Sigma_1\rangle^2] = \sum_{i=1}^{5}\left(\frac{h_2}{48}\varsigma_i^4 - \frac{\eta}{2}\varsigma_i^2\right), \quad (6.6)$$

where η is a Lagrange multiplier associated with the constraint

$$\mathrm{Tr}[\langle\Sigma_1\rangle^2] = M^2/2. \quad (6.7)$$

The first derivative of Eq. (6.6) is

$$\frac{h_2}{12}\varsigma_i^3 - \eta\varsigma_i = \varsigma_i\left(\frac{h_2}{12}\varsigma_i^2 - \eta\right), \quad \{i = 1, \dots, 5\}. \quad (6.8)$$

This will vanish for each ς_i provided either $\varsigma_i = 0$ or $\varsigma_i = \pm\varsigma_0$, where $\varsigma_0 \equiv \sqrt{12\eta/h_2}$. With regard to the sign of the nonzero ς_i, it can be resolved as we did in the Jordan frame [4]. Referring to Eq. (6.4), by means of a unitary transformation, we may interchange any negative element in $\langle\Sigma_1\rangle$ with the corresponding positive element in $-\langle\Sigma_1\rangle$. Thus, WLOG, we may assume that the elements of Σ_1 are nonnegative. There are then five distinct extrema, depending on the number k of zeros in Σ_1, so that $T_2 = 2(5-k)\varsigma_0^2 = M^2$, $k = \{0, 1, \dots, 4\}$. Therefore[15], $\varsigma_0 = \sqrt{1/(2(5-k))}\,M$, with corresponding Lagrange multiplier $\eta = h_2 M^2/(24(5-k)) > 0$.

To determine which of these five extrema are minima, we consider the second derivative is

$$\frac{\partial^2 V(\sigma)}{\partial\varsigma_i\partial\varsigma_j} = \delta_{ij}\left[\frac{h_2}{4}\varsigma_i^2 - \eta\right]. \quad (6.9)$$

This is diagonal as well (unlike the Jordan frame calculation [4]) with elements either $-\eta$ if $\varsigma_i = 0$ or $h_2\varsigma_0^2/4 - \eta$ if $\varsigma_i \neq 0$. Since $\eta > 0$, the extremum has an unstable mode if *any* of the ς_i is zero. Taking $k = 0$ then, we find that $V''(\varsigma_0) = h_2\varsigma_0^2/4 - \eta = h_2 M^2/40 - h_2 M^2/120 = h_2 M^2/60 > 0$, so this case is (locally) stable, just as before [4]. Hence,

$$\langle\Sigma_1\rangle = \varsigma_0 \mathbf{1_5}, \text{with } \varsigma_0 = M/\sqrt{10}. \quad (6.10)$$

In sum, we have confirmed that, in this model, the only possibility for SSB to a phase having a local minimum is $SO(10) \to SU(5)\otimes U(1)$. Having done so, we are now in a position to determine the masses of the vector bosons and the other heavy scalars arising from fluctuations in Σ.

[14] The factor $1/\sqrt{2}$ has been inserted so that $\{\mathcal{R}^a_1\}$ are the generators of $SU(5)\otimes U(1)$ with canonical normalization, $\mathrm{Tr}[\mathcal{R}^a_1\mathcal{R}^b_1] = \delta_{ab}/2$ for $a = \{1, 2, \dots, 25\}$. For further discussion, see Appendix B.

[15] ς_0 implicitly depends upon k, but we hope that will be clear in context without having to introduce more cumbersome notation.

VII. VECTOR BOSON MASSES

Another quantity that can be read directly from the Einstein frame action Eq. (5.8) is the mass of the vector bosons, which, in this section, will be shown to be $M_V = gM_P/\sqrt{5\xi}$. These masses arise from the scalars' covariant derivative in Eq. (5.8b)

$$\frac{1}{2}\mathrm{Tr}\,[D_\mu\Sigma]^2 = \frac{1}{2}\sum_a \left(\partial_\mu\sigma_a + gf^{abc}A^\mu_b\sigma_c\right)^2, \qquad (7.1)$$

using $\Sigma = \sqrt{2}\,\sigma_a R^a$, as in Sec. V. The f^{abc} are the structure constants for $SO(10)$. In the action, Eq. (5.8b), the field strength, $\mathrm{Tr}[F^2_{\mu\nu}]/4$, is canonically normalized. Therefore, Eq. (7.1) implies that the vector boson mass matrix is

$$(M_V^2)_{ab} = g^2 f^{acd} f^{bce}\langle\sigma_d\rangle\langle\sigma_e\rangle, \qquad (7.2)$$

As discussed in Appendix B, the 20 gauge bosons that acquire mass after SSB transform as conjugates $\mathbf{10_4}\oplus\overline{\mathbf{10}}_{\mathbf{-4}}$ of $SU(5)\otimes U(1)$. As a result, all 20 will have the same mass M_V, so we may simplify the calculation by summing

$$\mathrm{Tr}\big[\langle(M_V^2)\rangle\big] = g^2 C_G \textstyle\sum_a \langle\sigma_a\rangle^2 = g^2 C_G \mathrm{Tr}[\langle\Sigma\rangle^2], \qquad (7.3)$$

where C_G is the quadratic Casimir in the adjoint representation. In $SO(N)$, $C_G = (N-2)/2$, so

$$\mathrm{Tr}[(M_V^2)] = 4g^2\mathrm{Tr}[\langle\Sigma\rangle^2] = 4g^2M^2 = 4g^2M_P^2/\xi, \qquad (7.4)$$

where, in the last steps, we applied first the constraint Eq. (5.3) and then Eq. (5.11). As mentioned, the 20 particles have identical masses, so each of them has mass

$$M_V = gM_P/\sqrt{5\xi}, \qquad (7.5)$$

VIII. HEAVY SCALAR MASSES

Unlike the 25 massless vector bosons, the $SU(5)\otimes U(1)$ gauge symmetry does not protect the 25 corresponding adjoint scalars from acquiring invariant masses after SSB of $SO(10)$. Returning to the action, Eq. (5.8b), we have previously mentioned that formally it remains $SO(10)$ gauge invariant. As an aside, one might think it would permit $\langle\Sigma\rangle = 0$, but that is illusory, a result of using a nonlinear representation of the symmetry. As discussed in Sec. VI, the transformation to the Einstein frame tacitly requires $\langle\Sigma\rangle \neq 0$, and that property is also subsumed in the constraint conditions Eqs. (5.3), (6.7). Thus, despite appearances, $SO(10)$ must be spontaneously broken to arrive at Eq. (5.8b).

To determine the scalar masses, we shall start from the decomposition of Σ into block form, Eq. (6.2b). It is convenient to work in a gauge (*e.g.,* unitary gauge) in which the off-diagonal blocks involving Σ_2 have been "eaten" to give masses to the vector bosons, so that $\Sigma_2 = 0$. Then Σ takes the form:

$$\Sigma = \sqrt{2}\,\sigma_a R^a = \left(\begin{array}{c|c} \Sigma_1 & 0 \\ \hline 0 & -\Sigma_1^* \end{array}\right), \qquad (8.1)$$

with $\Sigma_1 = \sigma_a\mathcal{R}_1^a$. (*N.B.* Σ_1 is not diagonal.) As explained in Sec. V, Σ has only 44 independent DoF before SSB. With 20 absorbed by the vector bosons, only 24 DoF remain in Σ_1.

This results in a fundamental difference[16] between the masses of the $SU(5)\otimes U(1)$ scalar multiplet $\mathbf{24_0}$ associated with Σ or Σ_1 and the singlet $\mathbf{1_0}$ attributed to Ω through ζ.

We wish to solve the constraint conditions to make explicit the 24 DoF represented by Σ. Writing the adjoint field Σ in the form $\Sigma = \langle\Sigma\rangle + \Delta\Sigma$, we may expand the Einstein frame action Eq. (5.8) about the background $\langle\Sigma\rangle$, assumed as usual to be constant. Keeping only the terms depending on Σ, the Lagrangian density becomes

$$\mathcal{L}_S \equiv \frac{1}{2}\mathrm{Tr}\big[(D_\mu\Delta\Sigma)^2\big] + \frac{h_2}{96}\mathrm{Tr}\Big[\big(\langle\Sigma\rangle + \Delta\Sigma\big)^4\Big]. \qquad (8.2)$$

To determine the masses associated with $\Delta\Sigma$, we may neglect the gauge bosons in Eq. (8.2) and expand the potential terms through quadratic order in $\Delta\Sigma$. Recall from Sec. XI that $\langle\Sigma_1\rangle = \varsigma_0\mathbf{1_5}$, with $\varsigma_0 = M/\sqrt{10}$. Then Eq. (8.2) becomes

$$\begin{aligned}\mathcal{L}_S = \mathrm{Tr}\big[(\partial_\mu\Delta\Sigma_1)^2\big] + \frac{h_2}{48}\Big[5\varsigma_0^4 + \\ 4\varsigma_0^3\mathrm{Tr}[\Delta\Sigma_1] + 6\varsigma_0^2\mathrm{Tr}[\Delta\Sigma_1^2] + \dots\Big].\end{aligned} \qquad (8.3)$$

The normalization of the kinetic energy in Eq. (8.3) appears to be not canonical, but, from Eq. (8.1), $\Delta\Sigma_1 = \delta\sigma_a\mathcal{R}_1^a$ and $\mathrm{Tr}[\mathcal{R}_1^a\mathcal{R}_1^b] = \delta_{ab}/2$. Therefore,

$$\mathrm{Tr}[(\partial_\mu\Delta\Sigma_1)^2] = \frac{1}{2}\sum_1^{24}(\partial_\mu\delta\sigma_a)^2. \qquad (8.4)$$

We must now take into account the constraints Eqs. (5.3), (6.7),

$$\mathrm{Tr}\big[\big(\langle\Sigma_1\rangle + \Delta\Sigma_1\big)^2\big] = M^2/2, \text{ or} \qquad (8.5a)$$

$$2\varsigma_0\mathrm{Tr}[\Delta\Sigma_1] + \mathrm{Tr}[(\Delta\Sigma_1)^2] = 0. \qquad (8.5b)$$

To interpret this constraint, we decompose the 25 components of $\Delta\Sigma_1$ as

$$\Delta\Sigma_1 = \frac{\Delta S_1}{5}\mathbf{1_5} + \Delta\widetilde{\Sigma}_1, \qquad (8.6)$$

[16] The reader may wish to refer to the branching rules for $SO(10)\to SU(5)\otimes U(1)$, Eq. (B7).

with[17] $\mathit{\Delta} S_1 \equiv \mathrm{Tr}[\mathit{\Delta\Sigma}_1]$, so that $\mathrm{Tr}[\mathit{\Delta}\widetilde{\mathit{\Sigma}}_1] = 0$. Then Eq. (8.5) implies

$$2\varsigma_0 \mathit{\Delta} S_1 + \frac{1}{5}\mathit{\Delta} S_1{}^2 + \mathrm{Tr}\big[(\mathit{\Delta}\widetilde{\mathit{\Sigma}}_1)^2\big] = 0, \tag{8.7a}$$

$$\Longrightarrow \frac{\mathit{\Delta} S_1}{5\varsigma_0} = \sqrt{1 - 2\mathrm{Tr}[(\mathit{\Delta}\widetilde{\mathit{\Sigma}}_1)^2]/M^2} - 1 \approx \tag{8.7b}$$

$$-\mathrm{Tr}[(\mathit{\Delta}\widetilde{\mathit{\Sigma}}_1)^2]/M^2 + \mathrm{Tr}[(\mathit{\Delta}\widetilde{\mathit{\Sigma}}_1)^2]^2/(2M^4) + \ldots. \tag{8.7c}$$

In Eq. (8.7b), the positive square root must be chosen so that $\mathit{\Delta} S_1 \to 0$ for $\mathit{\Delta}\widetilde{\mathit{\Sigma}} \to 0$. The interpretation of Eq. (8.7) is that $\mathit{\Delta} S_1$ is determined by $\mathit{\Delta}\widetilde{\mathit{\Sigma}}_1$ of $SU(5)$, with the leading term of $\mathit{\Delta} S_1$ being quadratic in $\mathit{\Delta}\widetilde{\mathit{\Sigma}}_1$.

Returning to $\mathcal{L}_S$ in Eq. (8.3), we want to decompose $\mathit{\Delta\Sigma}_1$ as in Eq. (8.6) and replace $\mathit{\Delta} S_1$ using Eq. (8.7c). First, in the kinetic term, $(\partial_\mu \mathit{\Delta} S_1)^2$ can be discarded since it is actually fourth order in $\mathit{\Delta}\widetilde{\mathit{\Sigma}}_1$. Next, the second line of Eq. (8.3) can be reexpressed as

$$4\varsigma_0^3 \mathit{\Delta} S_1 + 6\varsigma_0^2 \mathrm{Tr}\big[(\mathit{\Delta}\widetilde{\mathit{\Sigma}}_1)^2\big] \approx 4\varsigma_0^2 \mathrm{Tr}\big[(\mathit{\Delta}\widetilde{\mathit{\Sigma}}_1)^2\big] + \ldots, \tag{8.8}$$

neglecting terms in $\mathit{\Delta}\widetilde{\mathit{\Sigma}}_1$ of higher order than quadratic.

Since the kinetic term is canonically normalized, the mass-squared of the 24 $SU(5)$ adjoint scalars is

$$M^2_{\mathit{\Delta\Sigma}} = 4\varsigma_0^2 = \frac{2}{5}M^2 = \frac{2}{5\xi}M_P^2 = \frac{2}{g^2}M_V^2, \tag{8.9}$$

where we have included the last two relations in order to facilitate comparison of these scalar masses with the Planck mass and the massive gauge bosons.

IX. THE BACKGROUND CURVATURE AND ITS FLUCTUATIONS

Renormalizable gravity is a scalar-tensor theory of gravity, *i.e.,* the metric involves a scalar DoF in addition to the usual tensor degree of freedom associated with the graviton. The scalar DoF can be identified with fluctuations of the scalar curvature $\widetilde{R}$. In the Jordan frame, we had to calculate the radiative corrections to the effective action to determine the magnitude of the background curvature $\langle R_J \rangle$ and its fluctuations. Therefore, it may come as a surprise that the first approximation to the corresponding quantity in the Einstein frame can be calculated from the classical action Eq. (5.8). On second thought, since Eq. (5.8b) contains a cosmological constant, both before and after SSB, one could have anticipated that $\widetilde{R} = 4\Lambda$ already from the matter action. Therefore, before getting into the matter of calculating radiative corrections to the effective action, we shall first discuss the tree approximation.

For this purpose, as well as to enable calculation of radiative corrections in the next section, we must make some simplifying assumptions. By defining $\Phi \equiv \Omega\mathit{\Sigma}$, we have distinguished the magnitude Ω of Φ from its direction $\mathit{\Sigma}$. At the classical level, Eq. (5.8), we saw that the two fields were essentially decoupled, *i.e.,* Ω is expressed through the dilaton field which does not couple directly to $\mathit{\Sigma}$. Consequently, we may replace $\mathit{\Sigma} \to \langle\mathit{\Sigma}\rangle$, while still keeping the background metric and ζ (or Ω) off-shell. In general, the metric $\widetilde{g}_{\mu\nu}$ couples directly to everything via the factor $\sqrt{\widetilde{g}}$ in the invariant volume density, but this will not cause problems in leading order. $\mathit{\Sigma}$ also couples to the metric through the kinetic term $(\partial_\mu \mathit{\Sigma})^2$ but, assuming that $\langle\mathit{\Sigma}\rangle$ is constant, $\partial_\mu\langle\mathit{\Sigma}\rangle = 0$.

Secondly, we shall assume that the background metric has maximal global symmetry, *viz.,* that of dS spacetime. In that case, the Euclidean spacetime volume is

$$\int d^4x \sqrt{\langle\widetilde{g}\rangle} = \left(\frac{12}{\widetilde{R}}\right)^2 \frac{8\pi^2}{3} \equiv \frac{V_4}{\rho^4}, \tag{9.1}$$

where, for economy of writing henceforth, we defined $\rho \equiv (\widetilde{R})^{1/2}$ and the unit volume $V_4 \equiv 12^2 \times 8\pi^2/3 = 384\pi^2$. This does not require ρ to be on-shell; it follows simply from the assumption that the background has maximal symmetry, so that the spacetime volume is a sphere S^4, with an arbitrary radius of curvature related to ρ.

Our goal is to determine the extrema of ρ and ζ and to determine which are minima. Our first task is to determine that they have stable, constant background fields. So for the moment, we shall assume that both are constant. With these assumptions, the classical action Eq. (5.8) is independent of ζ, since it only has derivative couplings. As we have explained in Sec. VI, this is because classically, ζ is a Goldstone boson. It is only from quantum corrections that we can determine whether ζ has a minimum, even for constant ζ. However, unlike the Jordan frame calculation that appears to only enable one to determine the ratio of fields[18], we can, as a consequence of the conformal transformation, determine the minimum in ρ directly from the classical action. Under the preceding assumptions, the value of the classical action Eq. (5.8) off-shell is

$$\frac{1}{V_4} S_{cl}^{(E)}(\rho) = \left[\frac{1}{3b} + \frac{c}{6} - \frac{\xi M^2}{2\rho^2} + \frac{h_3 M^4}{24\rho^4}\right]. \tag{9.2}$$

It may be surprising at first sight that the contributions from the higher- order action, Eq. (5.8a), are independent of ρ. It is clear that setting $\mathit{\Sigma} \to \langle\mathit{\Sigma}\rangle$ affects only Eq. (5.8b), but the curvature and dilaton fields enter Eq. (5.8a) as well. Upon reflection, this observation results from the

[17] Although $\mathit{\Delta} S_1$ transforms as a $SU(5) \otimes U(1)$ singlet, it must not be confused with the $\mathbf{1_0}$, $SO(10)$ singlet Ω (or ζ).

[18] In fact, because of the mixing between modes, we discovered in Ref. [4] only belatedly that the minimum in ρ called ε_1 was classical. The calculation in this section makes that clear from the outset.

assumption that the classical background fields have constant curvature ρ (or $\widetilde{R}$) and constant ζ (or Ω). Then the higher-derivative action, Eq. (5.8a), has the same classically scale invariant form as in the Jordan frame. Even off-shell, its value is independent of M, and, being dimensionless, it must also be independent of ρ.

To determine the extrema in ρ and its nature, we calculate the first two derivatives of Eq. (9.2):

$$\frac{1}{V_4}\frac{\partial S_{cl}^{(E)}}{\partial\rho} = \frac{\xi M^2}{\rho^3} - \frac{h_3 M^4}{6\rho^5}, \tag{9.3a}$$

$$\frac{1}{V_4}\frac{\partial^2 S_{cl}^{(E)}}{\partial\rho^2} = -\frac{3\xi M^2}{\rho^4} + \frac{5h_3 M^4}{6\rho^6}. \tag{9.3b}$$

Eq. (9.3a) vanishes for

$$\rho_0^2 = \frac{h_3}{6\xi}M^2 = \frac{h_3}{6\xi^2}M_P^2. \tag{9.4}$$

For $\rho = \rho_0$, the curvature Eq. (9.3b) of the potential becomes $2\xi M^2/\rho_0^4 = 2M_P^2/\rho_0^4 > 0$, so ρ_0 is in fact a minimum of the classical potential. To translate this into a mass parameter, we return to the Lagrangian density by dividing the action by the invariant spacetime volume Eq. (9.1). Expanding $\rho = \rho_0 + \delta\rho$ to second order in $\delta\rho$,

$$S_{cl}^{(E)}(\rho) = S_{cl}^{(E)}(\rho_0) + \int d^4x\sqrt{\widetilde{g}}\;\frac{m_\rho^2}{2}\,\delta\rho^2 + \ldots, \tag{9.5a}$$

$$\text{with } m_\rho \equiv \sqrt{2}\,M_P. \tag{9.5b}$$

For future reference, the on-shell value of the classical action in the Einstein frame, Eq. (9.2), is

$$\frac{1}{V_4}S_{cl}^{(E)}(\rho_0) = \left[\frac{1}{3b} + \frac{c}{6} - \frac{\xi M^2}{4\rho_0^2}\right] = \left[\frac{1}{3b} + \frac{c}{6} - \frac{3\xi^2}{2h_3}\right]. \tag{9.6}$$

Eq. (9.5) is only valid for constant fluctuations $\delta\rho$; however, we need to demonstrate stability for non-static fluctuations of the background. This becomes complicated, even assuming that the background is dS spacetime with constant curvature $\widehat{R} = \rho_0^2$. So long as we assume that the background ζ is constant, this analysis can be carried out classically by expanding Eq. (5.8) to second-order in metric fluctuations. To express local fluctuations in the metric, we follow the same path as in Sec. III, writing $\widetilde{g}_{\mu\nu} \equiv \widehat{g}_{\mu\nu} + h_{\mu\nu}$, with the background $\widehat{g}_{\mu\nu}$ describing dS spacetime with constant curvature $\widehat{R} = \rho_0^2$, and $h_{\mu\nu}(x)$ corresponding to the fluctuations. For $h_{\mu\nu}(x)$, we adopt the transverse-traceless (TT) gauge, described in Eq. (3.6) *et seq.* To explore stability, we need to expand the fluctuations through second order, up to which there is no mixing between the fluctuations of fields having nontrivial classical backgrounds and those that do not. The dilaton field is exceptional, inasmuch as it only appears in Eq. (5.8) derivatively coupled. In that case, its fluctuations still do not mix with other fields to quadratic order. Assuming that the background vector field A_μ vanishes, the result for fluctuations to the metric will, to quadratic order, be the same *as if* we started from the classical action

$$S_{ho}^{(E)} = \int d^4x\sqrt{\widetilde{g}}\left[\frac{1}{2a}\widetilde{C}^2 + \frac{1}{3b}\widetilde{R}^2 + c\,\widetilde{G}\right], \tag{9.7a}$$

$$S_m^{(E)} = \int d^4x\sqrt{\widetilde{g}}\left[-\frac{\xi M^2}{2}\widetilde{R} + \frac{h_3}{24}M^4\right], \tag{9.7b}$$

This is precisely the action for renormalizable gravity with the inclusion an explicit Planck mass, Eq. (5.11) and a cosmological constant, Eq. (5.12). This may be obtained from the model discussed in Sec. III for the real field, Eq. (3.1) with the replacements

$$\delta\phi \to 0, \quad \varphi \to M, \quad \lambda \to h_3/6. \tag{9.8}$$

This model was previously analyzed by Avramidi [13]. As we mentioned in the Introduction, Sec. I, by expanding in the Jordan frame, he showed that, with the exception of the five zero modes that we discussed earlier, the fluctuations are stable for a certain range of coupling constants, a result that seems not to be as well known as perhaps it should be. This conclusion should apply to the Einstein frame on-shell, since the difference in the actions between the two frames is simply a field redefinition[19]. Therefore, we can simply adapt Avramidi's results[20] to the action, Eq. (9.7). We must have $a, b > 0$ and

$$\text{tensor: } \frac{2b}{3a} < 1 + \frac{3\xi^2 a}{h_3}; \quad \text{scalar: } 18\xi^2 < \frac{h_3}{b}. \tag{9.9}$$

This calculation has been regarded as purely classical. When quantum corrections are calculated, these couplings become running couplings, and these inequalities must be respected at a certain symmetry-breaking scale v that will be defined precisely in the next section. We wrote these inequalities in a form that takes advantage of the fact that $\xi(\mu)$ and the ratios of couplings $b(\mu)/a(\mu), h_3(\mu)/a(\mu)$ approach finite UVFPs as $\mu \to \infty$, so it is most convenient to study their running in the range of scales above v, as we did in Ref. [4]. We now turn to the determination of the quantum corrections to the effective action.

X. SCALE OF SYMMETRY BREAKING AND THE DILATON MASS

The developments in the preceding sections all stemmed from the supposition that

$$\langle\Phi\rangle = \langle\Omega\rangle\langle\mathit{\Sigma}\rangle \neq 0, \tag{10.1}$$

which permitted transformation to the Einstein frame. In that frame, unlike the Jordan frame, we were able to

[19] For further discussion, see *e.g.,* Ref. [36].
[20] See Eqs. (4.170), (4.171) of Ref. [13].

identify the classical values of the Planck mass, M_P, the cosmological constant Λ, $\langle\Sigma\rangle$,$\left\langle\widetilde{R}\right\rangle$ as well as the masses for all the fields except for the dilaton ζ, which classically appears as a free, massless scalar[21]. In this section, we wish to determine $\langle\zeta\rangle$, or equivalently $\langle\Omega\rangle$, and the dilaton mass m_d, by giving the radiative corrections to the effective action.

In general, the analytic calculation of radiative corrections to the effective action is impossible, and it has seldom been done for any spacetime-dependent background $\langle R(x)\rangle$ or $\langle\Phi(x)\rangle$. (This is also true for fields in flat spacetime, with instantons being an exception [37].) For backgrounds having $\langle R\rangle$ and $\langle\Phi\rangle$ spacetime independent, the one-loop corrections can be performed; even then, only bits and pieces of the two-loop corrections have been calculated to date.

Turning to the dilaton field ζ, the classical action Eq. (5.8) depends on ζ only through its gradient $\nabla_\mu\zeta = \partial_\mu\zeta$, reflecting its role as a Goldstone boson associated with scale breaking $\langle\Omega\rangle \neq 0$. As remarked earlier, classical scale invariance is explicitly broken in the QFT, and the dilaton will get a nonzero mass m_d at two-loop order. To determine whether or not it represents an instability, we shall have to calculate these radiative corrections. As mentioned in Sec. III, in our earlier work [4] in the Jordan frame, we showed that the two-loop corrections responsible for $m_d \neq 0$ could be calculated knowing only the one-loop β-functions. We also learned that $m_d^2 > 0$ for some range of couplings, but, being unsure of the proper normalization of the dilaton field, we could only determine m_d within a multiplicative factor. Here, we wish to confirm those results and to determine the dilaton mass m_d more precisely.

In order to be able to compare with our previous work [4], we begin with the form given there for the effective action in the Jordan frame in dS background:

$$\frac{\Gamma^{(J)}}{V_4} = \frac{S_{cl}^{(J)}(r)}{V_4} + \frac{1}{2}B(r)\log\frac{R_J}{\mu^2} + \frac{1}{8}C(r)\left(\log\frac{R_J}{\mu^2}\right)^2 + \ldots, \tag{10.2}$$

where, we recall, the ratio $r \equiv T_2/R_J$, and V_4 has been defined earlier below Eq. (9.1). This presumes that both Φ and R_J are constant. For constant Φ, the transformation from Jordan to the Einstein frame yields $R_J \to \Omega^2\widetilde{R}$ and $r \to M^2/\widetilde{R}$, independent of Ω. With $\rho \equiv (\widetilde{R})^{1/2}$, defined beneath Eq. (9.1),

$$\log\frac{R_J}{\mu^2} \to 2\left(\frac{\zeta}{\sqrt{6\xi'}M} + \log\frac{\rho}{\mu}\right). \tag{10.3}$$

In the Jordan frame, we thought of the one-loop corrections as bringing in the dependence on the scalar curvature R_J for a fixed ratio r. By contrast, in the Einstein frame, fixed r represents fixed scalar curvature, $\widetilde{R}$, and the dependence on the dilaton field ζ enters through the corrections.

The effective action, like the classical action, is dimensionless, so it is not rescaled or changed by a conformal transformation. Therefore, Eq. (10.2) becomes

$$\begin{aligned}\frac{\Gamma^{(E)}}{V_4} = \frac{S_{cl}^{(E)}\left(\frac{M^2}{\rho^2}\right)}{V_4} + B\left(\frac{M^2}{\rho^2}\right)\left(\frac{\zeta}{\sqrt{6\xi'}M} + \log\frac{\rho}{\mu}\right) + \\ \frac{1}{2}C\left(\frac{M^2}{\rho^2}\right)\left(\frac{\zeta}{\sqrt{6\xi'}M} + \log\frac{\rho}{\mu}\right)^2 + \ldots.\end{aligned} \tag{10.4}$$

The original scale invariance is still reflected indirectly in Eq. (10.4) by the property that, in the parentheses involving $\log(\rho/\mu)$, changing the normalization scale μ can be offset by a shift in ζ. Thus, although ζ is no longer derivatively coupled when radiative corrections are included, its value $\langle\zeta\rangle$ is not renormalization group invariant, and, therefore, not directly observable. We shall exploit this shift freedom shortly.

All dependence on ζ in the effective action, Eq. (10.4), enters through the radiative corrections. The first derivative is

$$\begin{aligned}\frac{1}{V_4}\frac{\partial\Gamma^{(E)}}{\partial\zeta} = \left(\frac{1}{\sqrt{6\xi'}M}\right)\Bigg[B\left(\frac{M^2}{\rho^2}\right) + \\ C\left(\frac{M^2}{\rho^2}\right)\left(\frac{\zeta}{\sqrt{6\xi'}M} + \log\frac{\rho}{\mu}\right)\Bigg].\end{aligned} \tag{10.5}$$

To one-loop order, $B \to B_1$ and $C \to 0$. To have an extremum in ζ, therefore, it must be that $B_1(M^2/\rho^2) = 0$. To this order, we may replace ρ by its classical value, $\rho_0 = \sqrt{h_3/(6\xi)}M$ from Eq. (9.4), so the extremum in ζ is determined by the equation

$$B_1\big(6\xi(\mu)/h_3(\mu)\big) = 0. \tag{10.6}$$

This equation is not true for all choices of μ. Its interpretation is that, in order for perturbative DT to occur, we must be able to find a scale, $\mu = v$, at which this relation among couplings holds. In previous work in Jordan frame [4], we have obtained an explicit formula for $B_1(r)$, and Eq. (10.6) is in fact identical to the Jordan frame condition for an extremum in R_J at fixed r. We showed that this equation can be satisfied for a range of coupling constants within the basin of attraction of the UVFP.

Because Eq. (10.6) is independent of ζ, we must go beyond one-loop order to determine a nonzero mass for the dilaton. Even at one-loop order, however, we expect the classical minimum in the curvature at ρ_0 to change slightly, $\rho_0 \to \rho_0 + \delta\rho_0$, to which end we calculate the first

---

[21] In the Jordan frame, had we made the assumption Eq. (10.1), we *could* have performed an expansion about that background, but it is still simpler to do in the Einstein frame with no nonminimal coupling(s) to R.

derivative of the effective action with respect to ρ:

$$
\begin{aligned}
\frac{1}{V_4}\frac{\partial\Gamma^{(E)}}{\partial\rho} &= \frac{1}{V_4}\frac{\partial S^{(E)}_{cl}}{\partial\rho} + \frac{1}{\rho}B\left(\frac{M^2}{\rho^2}\right) \\
&- \frac{2M^2}{\rho^3}B'\left(\frac{M^2}{\rho^2}\right)\left(\frac{\zeta}{\sqrt{6\xi'}M} + \log\frac{\rho}{\mu}\right) + \\
&\frac{1}{\rho}C\left(\frac{M^2}{\rho^2}\right)\left(\frac{\zeta}{\sqrt{6\xi'}M} + \log\frac{\rho}{\mu}\right) + \ldots,
\end{aligned}
\tag{10.7}
$$

where we truncated the equation for reasons to be explained below. As with Eq. (10.5), the one-loop correction has $B{\to}B_1$ and $C{\to}0$. It is convenient to choose the normalization scale $\mu = v$, at which Eq. (10.6) holds, so that the second term on the RHS in Eq. (10.7) vanishes. Then, to first order, we expand in $\delta\rho_0$ to get

$$
\begin{aligned}
\frac{1}{V_4}\frac{\partial\Gamma^{(E)}}{\partial\rho} &\approx \frac{1}{V_4}\frac{\partial^2 S^{(E)}_{cl}}{\partial\rho^2}\bigg|_{\rho_0}\delta\rho_0 - \\
&\frac{2M^2}{\rho_0^3}B_1'\left(\frac{M^2}{\rho_0^2}\right)\left(\frac{\zeta}{\sqrt{6\xi'}M} + \log\frac{\rho_0}{v}\right),
\end{aligned}
\tag{10.8a}
$$

$$
\approx \frac{2M^2}{\rho_0^3}\left[\frac{\xi\delta\rho_0}{\rho_0} - B_1'\left(\frac{M^2}{\rho_0^2}\right)\left(\frac{\zeta}{\sqrt{6\xi'}M} + \log\frac{\rho_0}{v}\right)\right]. \tag{10.8b}
$$

Given M, setting Eq. (10.8b) to zero and $\zeta{\to}\langle\zeta\rangle$ determines a relation between $\delta\rho_0$ and $\langle\zeta\rangle$. The value of $\langle\zeta\rangle$ is still not fixed at one-loop order, as we pointed out earlier. Therefore, we may conveniently choose $\langle\zeta\rangle$ such that

$$
\langle\zeta\rangle + \sqrt{6\xi'(v)}\,M\log\frac{\rho_0}{v} = 0, \tag{10.9}
$$

where $\xi'(v) \equiv \xi(v){+}1/6$, and ρ_0 is given in Eq. (9.4). With this choice for $\langle\zeta\rangle$, we may conclude that the first-order correction $\delta\rho_0$ vanishes!

Although we have a one-loop constraint Eq. (10.6) consistent with ζ having a local extremum, we have not determined its character. To do so requires going to two-loop order. Although not all two-loop corrections to the effective action or to the β-functions are known, some two-loop effects are calculable from the one-loop β-functions [38], including $C_2(r)$, the first nonzero contribution to $C(r)$. Fortunately, these turn out to be sufficient to determine the two-loop contributions to the effective action that are required [4].

To see that in the present language, we need the second variations which, on-shell with our conventions for M and $\langle\zeta\rangle$, take the form

$$
\frac{1}{V_4}\frac{\partial^2\Gamma^{(E)}}{\partial\zeta^2}\bigg|_{os} = \frac{1}{6\xi' M^2}C_2\left(\frac{M^2}{\rho_0^2}\right), \tag{10.10a}
$$

$$
\begin{aligned}
\frac{1}{V_4}\frac{\partial^2\Gamma^{(E)}}{\partial\zeta\partial\rho}\bigg|_{os} &= \left(\frac{1}{\sqrt{6\xi'}M}\right)\left[-\frac{2M^2}{\rho_0^3}B_1'\left(\frac{M^2}{\rho_0^2}\right) + \right. \\
&\left. \frac{1}{\rho_0}C_2\left(\frac{M^2}{\rho_0^2}\right)\right],
\end{aligned}
\tag{10.10b}
$$

$$
\begin{aligned}
\frac{1}{V_4}\frac{\partial^2\Gamma^{(E)}}{\partial\rho^2}\bigg|_{os} &= \frac{1}{V_4}\frac{\partial^2 S^{(E)}_{cl}}{\partial\rho^2}\bigg|_{\rho=\rho_0} - \frac{2}{\rho_0^3}B_1'\left(\frac{M^2}{\rho_0^2}\right) + \\
&\frac{1}{\rho_0}C_2\left(\frac{M^2}{\rho_0^2}\right) + \ldots,
\end{aligned}
\tag{10.10c}
$$

where the subscript os refers to the value "on-shell." Having arranged for the one-loop correction to ρ_0 to vanish, this means $\rho{\to}\rho_0, \zeta{\to}\langle\zeta\rangle$, with $\langle\zeta\rangle$ given by Eq. (10.9). In the last line, Eq. (10.10c), we have omitted certain other one- and two-loop contributions for reasons that will become clear shortly. Consider the matrix of second variations

$$
\delta^{(2)}\Gamma^{(E)} = \frac{1}{2}\begin{pmatrix}\delta\zeta & \delta\rho\end{pmatrix}\begin{bmatrix}\dfrac{\partial^2\Gamma^{(E)}}{\partial\zeta^2} & \dfrac{\partial^2\Gamma^{(E)}}{\partial\zeta\partial\rho} \\[2ex] \dfrac{\partial^2\Gamma^{(E)}}{\partial\zeta\partial\rho} & \dfrac{\partial^2\Gamma^{(E)}}{\partial\rho^2}\end{bmatrix}\begin{pmatrix}\delta\zeta \\ \delta\rho\end{pmatrix}. \tag{10.11}
$$

To review the order of the matrix elements, we recall that the leading nonvanishing term of Eq. (10.10a) is $O(\hbar^2)$, *i.e.,* two loops; of Eq. (10.10b), $O(\hbar)$; of Eq. (10.10c), $O(1)$. Thus, the matrix has a familiar "see-saw" pattern, the same structure that was encountered in the Jordan frame calculation [4]. The determinant is $O(\hbar^2)$ and the trace is $O(1)$, so one eigenvalue is $O(1)$ and the other $O(\hbar^2)$.

Naturally, the larger one is associated with the classical fluctuation determined in the previous section. To be precise, we take the classical approximation for $\partial^2\Gamma^{(E)}/\partial\rho^2$, *viz.,* $2\xi M^2/\rho_0^4$ from just below Eq. (9.4). Then the larger eigenvalue of the matrix in Eq. (10.11)

$$
\varepsilon_1 = \frac{2\xi M^2 V_4}{\rho_0^4} + O(\hbar^2), \tag{10.12}
$$

with eigenvector $(\delta\zeta,\delta\rho){=}(0,1){+}O(\hbar^2)$. When divided by the spacetime volume V_4/ρ_0^4, ε_1 gives precisely the value m_ρ^2 in Eq. (9.5). Having arranged for the one-loop correction to m_ρ to vanish, we are not really interested in its two-loop corrections. In fact, they would require B_2, which is not known and cannot be determined using one-loop β-functions. It would also require taking into account gravitational corrections to the wave-function renormalization.

The smaller eigenvalue ε_2 is associated with the dilaton,

$$
\varepsilon_2 = \left(\frac{1}{16\pi^2}\right)^2\frac{V_4}{6\xi' M^2}\left[C_2 - \frac{B_1'^2}{2\xi}\right] + O(\hbar^3), \tag{10.13}
$$

with eigenvector

$$(\delta\zeta,\delta\rho)=(1,-\sqrt{h_3/(\xi'\xi^3)}B_1'/6)+O(\hbar^2). \qquad (10.14)$$

All scale-dependent quantities on the RHS of Eqs. (10.13), (10.14) are to be evaluated at the DT scale $\mu=v$, where Eq. (10.6) is fulfilled. We have made explicit the factors of $16\pi^2$, heretofore suppressed, in order to emphasize how very much smaller than ε_1 this is. If we divide by the spacetime volume, we find

$$\frac{m_d^2}{M_P^2}=\left(\frac{1}{16\pi^2}\right)^2\frac{1}{30\xi\xi'}\left(\frac{h_3}{6\xi}\right)^2\left[C_2-\frac{B_1'^2}{2\xi}\right]\Bigg|_{\mu=v}, \qquad (10.15)$$

corresponding to a term in the effective action

$$\int d^4x\sqrt{\widetilde{g}}\;\frac{m_d^2}{2}\,\delta\zeta^2+O(\hbar^3). \qquad (10.16)$$

Are we really justified in identifying this with the dilaton mass? We believe the answer is yes, although it does require further justification. The fact that the eigenvalue Eq. (10.13) is already of $O(\hbar^2)$ hides a multitude of sins of omission. For example, we did not address the one-loop corrections to the spacetime volume[22], but that is clearly not necessary in order to determine ε_2 through $O(\hbar^2)$ in Eq. (10.13). Similarly, any mixing of $\delta\zeta$ with $\delta\rho$ affects m_d^2 in $O(\hbar^3)$ or higher.

There is also the related issue of whether the kinetic term (wave function normalization) for $\delta\zeta$ is canonical. The preceding calculations assumed the fluctuations $(\delta\zeta,\delta\rho)$ were constant, but we must go beyond the static limit to answer this question. In fact, a glance at the original Einstein-frame action Eq. (5.8) casts doubt on this. Besides the canonical term for ζ in Eq. (5.8b), there are also the higher-derivative terms in Eq. (5.8a). Since they are classical, *i.e.,* $O(\hbar^0)$, they cannot be ignored in general.

We propose to deal with them as follows: As will be discussed further in Sec. XI, below the Planck scale M_P, the gravitational theory is well-approximated by the E-H action plus higher-dimensional operators. This is in effect a derivative expansion in $1/M_P$. Since $m_d \ll M_V \lesssim v$, we may consider an expansion in $1/v$ on momentum scales small compared to all the particles that acquire masses after SSB in tree approximation in Einstein frame, which were discussed in Sections VII, VIII, and IX. In that case, all the terms in Eq. (5.8a) involving ϑ_μ comprise operators of higher dimension than four and, thus, will be small compared with the terms remaining. Then the leading contribution to the kinetic term for ζ is entirely from Eq. (5.8b), which is simply the canonical term $(\widetilde{\nabla}\zeta)^2/2$. In that case, what we have called m_d^2 in Eq. (10.15) above is in fact correct. Furthermore, since $-\widetilde{\Box}$ is an elliptic operator on Hilbert space, we may conclude that the non-static fluctuations in this approximation will also be stable.

XI. LOW-ENERGY EFFECTIVE FIELD THEORY

There have been several physical scales variously identified as $M_P, \Lambda, M_V, M_{\Delta\Sigma}, m_\rho$, as well as v and m_d, their ratios are in principle observables. Unfortunately, all but m_d, are likely to be $O(M_P)$, although we have not exhaustively explored the range of parameter space delineated by the determination of v, Eq. (10.6), and the requirement that $\varepsilon_2 > 0$, Eq. (10.13), or, equivalently, $m_d^2 > 0$, Eq. (10.15). As with superstring phenomenology, the only natural realm of application at such scales is to precision cosmology around the time of the Big Bang and earlier. On the other hand, unlike superstring theory, QFT can deal with the time evolution of (gauge-invariant) correlation functions, provided the measurement frame is specified. This calls attention to the issue of whether renormalizable gravity is unitary at scales above v. We expect to have more to say about this in the future, but we will have little to contribute to the debate in this paper.

Near the end of the previous section, we argued that there may be a range of momentum scales, $m_d < p \lesssim M_V \lesssim M_P$ in which all particles except the massless vector bosons of $SU(5)\otimes U(1)$, the massless graviton, and the dilaton have become irrelevant. The corresponding low-energy, classical action can be extracted from Eq. (5.8) with the inclusion of the dilaton mass term Eq. (10.16):

$$\begin{aligned} S^{(E)}_{eff}=&\int d^4x\sqrt{\widetilde{g}}\left[\frac{1}{2a}\widetilde{C}^2+\frac{1}{3b}\widetilde{R}^2+c\,\widetilde{G}+\frac{h_3}{24}M^4\right.\\ &\left.-\frac{\xi M^2}{2}\widetilde{R}+\frac{1}{4}\mathrm{Tr}[\widetilde{F}^2_{\mu\nu}]+\frac{(\widetilde{\nabla}\delta\zeta)^2}{2}+\frac{m_d^2\,\delta\zeta^2}{2}\right], \end{aligned} \qquad (11.1)$$

where $\widetilde{F}_{\mu\nu}$ represents the $SU(5)\otimes U(1)$ field strength for the massless gauge bosons. Recalling Eqs. (5.9), (5.11), we have neglected terms involving $\theta_\mu=\sqrt{\xi/6\xi'}\,\widetilde{\partial}_\mu\zeta/M_P$, with $\xi'=\xi+1/6$, since, for this range of energy scales, these terms are of the same order as others dropped. Only the dilaton and the massless vectors of $SU(5)\otimes U(1)$ remain in addition to the metric $\widetilde{g}_{\mu\nu}$. We could also calculate some of the higher-dimensional operators that have been neglected, but they are not of great interest for present purposes unless the low-energy, effective action based on Eq. (11.1) proved to be unstable or to have zero-modes that may be removed by such higher-order terms.

The on-shell solution for the background turns out to correspond to constant curvature $\widehat{R}=4\Lambda$, with $\Lambda=h_3M_P^2/(24\xi^2)$. Assuming that the background has dS global symmetry, as in our earlier discussions, we can expand about the background to explore stability. It will

[22] Of course, we should have raised the same point below Eq.(10.12). Fortuitously, having arranged in Eq.(10.9) for $\delta\rho_0=0$ in $O(\hbar)$, such corrections to Eq. (10.12) will be at least $O(\hbar^2)$.

come as no surprise that the fluctuations will be stable, since we require $m_d^2 > 0$. There will remain the by-now familiar five conformal, zero modes associated with coherent fluctuations about S^4 background.

XII. CONCLUSIONS

In Ref. [4], we discussed a classically scale-invariant model in which renormalizable gravity is coupled to matter in the form of an $SO(10)$ gauge field plus a real scalar field in the adjoint representation. We showed that the model contains a locally stable UVFP, so that all couplings are AF. Moreover, the domain of attraction of the UVFP includes a region of parameter space corresponding to spontaneous breaking of the gauge symmetry to $SU(5)\otimes U(1)$, with the scalar multiplet acquiring a VEV. This VEV is perturbatively determined and calculable by DT, which determines the scale at which a specific relationship among the various dimensionless couplings holds true, and its presence generates an E-H term from the nonminimal scalar coupling to gravity. The quartic behavior of the metric's propagator may not admit an ordinary particle interpretation, but it is not an obstacle to the calculation of Euclidean correlation functions.

In this paper, the same model was transformed from Jordan to the Einstein frame, and the form of the one-loop effective action there was further developed. The Planck mass, cosmological constant, vector boson masses, and related scales were unambiguously identified. In particular, we were able to identify the canonically normalized dilaton field ζ and to determine the dilaton-mass, m_d, Eq. (10.15). We wish to re-emphasize that, even though the mass is a two-loop effect, it can be calculated knowing only the one-loop beta-functions. Whether such a light dilaton plays an important role in cosmological applications remains to be determined in future, as do other issues such as inflation and Dark Energy.

We showed that the effective field theory below the scale of symmetry breaking takes the form of the gauged $SU(5)\otimes U(1)$ nonlinear sigma model plus a dilaton and graviton. Of course, we would like this to be prototypical of a realistic model; obviously much remains to be done with regard to demonstrating a realistic SM-like theory at low energies, including in particular the emergence of the electroweak scale.

Although we have likened our determination of the symmetry-breaking scale dynamics to DT à la Coleman-Weinberg [12], we wish to reemphasize certain differences from their mechanism. In their seminal treatment, the self-coupling $\lambda(\mu)$ of the scalar field is unusually small in a neighborhood of the DT scale $\mu = v$. Indeed, $\lambda(v)$ is of the same order as the one-loop amplitude, $O(\alpha^2)$, very near to where $\lambda(\mu) = 0$.

In our application, the picture is different and is in fact frame dependent. In the Jordan frame [4], which is most nearly similar to Ref. [12], we first determined the direction of symmetry-breaking and the ratio $\langle\Phi\rangle/\rho$, where $\rho \equiv \sqrt{\langle R_J\rangle}$, from extremizing the classical potential. We then determined the value of the scalar curvature $\rho = v$ from the radiative corrections. In a neighborhood of this scale v, the one-loop correction to the effective action $\Gamma^{(J)}$, Eq. (10.2), becomes unusually small, of order of the two-loop correction. More precisely, we seek the value of ρ at which

$$\rho \frac{\partial \Gamma^{(J)}}{\partial \rho}\bigg|_{\rho=v} = B_1(\mu) + B_2(\mu) + C_2(\mu) \log \frac{v}{\mu} = 0. \quad (12.1)$$

If we choose the normalization scale $\mu = v$, Eq. (12.1) simplifies to $B_1(v)+B_2(v) = 0$. Thus, at the extremum, the one-loop correction B_1 is of order of the two-loop correction $B_2(v)$. In first approximation, the extremum occurs where $B_1(v) = 0$, a relation among couplings at scale v. In short, as compared with DT in Ref. [12], our application is of higher order in the loop-expansion. Instead of the extremum occurring at the scale v where a tree coupling λ falls to $O(\hbar)$ corrections, our extremum v is determined by the scale at which the $O(\hbar)$-correction falls to $O(\hbar^2)$. The determination that the extremum is in fact a minimum is a two-loop effect which, fortunately, was calculable from the one-loop β-functions.

In the Einstein frame, Eq. (5.8), the story was rather different, although the results were the same. Since only the $\Omega(x)$ field was used in performing the conformal transformation, the calculation in Sec. XI of the "directions" $\langle\Sigma\rangle$ at which extrema occur was essentially the same as before in Ref. [4], as was the determination of which one was a local minimum. Unlike the Jordan frame calculation, we were able to determine a first approximation to the scalar curvature $\rho \equiv (\widetilde{R})^{1/2}$ and to show it was a local minimum already in tree approximation, Eq. (9.4). In contrast, the dilaton degree of freedom ζ enters the effective potential Eq. (10.4) only via radiative corrections, and, we found the DT equation $B_1(v) = 0$ as a result of seeking the extremum in ζ. We were also able to calculate the one-loop correction $\delta\rho_0$ to the curvature ρ_0, Eq. (10.8b), and, by a propitious choice for $\langle\zeta\rangle$, we arranged for it to vanish. We then were able to calculate the curvature in ζ and thereby determine the dilaton mass m_d, Eq. (10.15), something which we had only be able to estimate previously.

In the original description in the Jordan frame, it was clear that the metric has a scalar DoF, *i.e.,* that this is a scalar-tensor theory of gravity. In the Einstein frame, this DoF was represented by the conformal field ρ in Sec. X. In the low energy effective field theory, Sec. XI, this scalar DoF does not appear, *i.e.,* it decouples (except for the five zero modes.) Even though the dilaton mass is proportional to the scale of SSB, it is a two-loop effect and $m_d^2/M_P^2 \ll 1$, Eq. (10.15). Unlike the other massive scalars, it does involve mixing with the scalar DoF of the metric.

These conclusions do not depend in detail on this particular model, and we expect them to be generic. While that is hopeful for finding a renormalizable extension of the SM to include gravity, it also suggests that it may be very difficult to test experimentally.

We have not discussed analytic continuation from Euclidean to Lorentzian signature. We simply assumed that for relevant spacetimes, it can be performed. New issues arise however: not even dS remains compact, although, depending on the frame, it is often the case that a fixed time slice has compact spatial volume. Although correlation functions remain well-defined, they can become IR divergent as the timelike separation between spacetime points grows indefinitely. This further complicates the discussion of unitarity, but in the past, all such perturbative infrared divergences in QFT have been resolved by a careful specification of observables. Regardless, having settled the primary issues of instability and ghosts that caused this line of investigation to be abandoned nearly 40 years ago, we are optimistic that eventually asymptotically free models based on renormalizable gravity will turn out to be consistent, unitary completions of Einstein-Hilbert gravity. Whether they can be extended to include the SM fields while preserving naturalness down to the electroweak scale remains a theoretical challenge.

ACKNOWLEDGMENTS

This research was supported in part by the National Science Foundation under Grant No. PHY11-25915 (KITP) and by the Baggs bequest (Liverpool). One of us (MBE) would like to thank Arkady Tseytlin for discussions.

Appendix A: Conformal Redefinition of Metric

In this Appendix, we summarize some formulas associated with conformal transformations of the metric[23]. As in the text, we choose Euclidean signature and choose the definition of the Ricci tensor so that $R > 0$ for positive curvature. Given the definitions, Eq. (4.1), $\widetilde{g}_{\mu\nu} \equiv \Omega^{-2} g_{\mu\nu}$, then obviously $\sqrt{\widetilde{g}} = \Omega^{-4}\sqrt{g}$, or in n-dimensions, $\sqrt{\widetilde{g}} = \Omega^{-n}\sqrt{g}$.

Relations among conformally-related curvatures are often more simply expressed when written in terms of $\Upsilon \equiv \log(\Omega)$, and here we state the results in terms of Υ instead of Ω.

Some useful identities are

$$\Omega^{-1}\nabla_\mu\Omega = \nabla_\mu\Upsilon, \tag{A1a}$$

$$\Omega^{-1}\nabla_\mu\nabla_\nu\Omega = \nabla_\mu\Upsilon\,\nabla_\nu\Upsilon + \nabla_\mu\nabla_\nu\Upsilon, \tag{A1b}$$

$$\Omega\nabla_\mu\nabla_\nu\Omega^{-1} = \nabla_\mu\Upsilon\,\nabla_\nu\Upsilon - \nabla_\mu\nabla_\nu\Upsilon. \tag{A1c}$$

[23] With appropriate adjustments for sign conventions, our formulae agree with Appendix G of Ref. [39].

The conformal transform of the connection is

$$\Gamma^\kappa_{\mu\nu} = g^{\kappa\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})/2 \to \tag{A2a}$$

$$\begin{aligned}\widetilde{\Gamma}^\kappa_{\mu\nu} &= \Gamma^\kappa_{\mu\nu} - \Delta^\kappa_{\mu\nu}, \text{ with}\\ \Delta^\kappa_{\mu\nu} &\equiv \delta^\kappa_\mu\partial_\nu\Upsilon + \delta^\kappa_\nu\partial_\mu\Upsilon - g_{\mu\nu}g^{\kappa\lambda}\partial_\lambda\Upsilon.\end{aligned} \tag{A2b}$$

$\Delta^\kappa_{\mu\nu}$ transforms as an ordinary tensor under general coordinate transformations. The Riemann curvature, $R^\kappa{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\kappa_{\lambda\nu} - \partial_\nu\Gamma^\kappa_{\lambda\mu} + \Gamma^\kappa_{\mu\rho}\Gamma^\rho_{\lambda\nu} - \Gamma^\kappa_{\nu\rho}\Gamma^\rho_{\lambda\mu}$, transforms as

$$\begin{aligned}\widetilde{R}^\kappa{}_{\lambda\mu\nu} = R^\kappa{}_{\lambda\mu\nu} + \delta^\kappa_{[\mu}\nabla_{\nu]}\vartheta_\lambda + \nabla^\kappa\vartheta_{[\mu}g_{\nu]\lambda} +\\ \delta^\kappa_{[\mu}\vartheta_{\nu]}\vartheta_\lambda - \vartheta^2\delta^\kappa_{[\mu}g_{\nu]\lambda},\end{aligned} \tag{A3}$$

where $\vartheta_\mu \equiv \partial_\mu\Upsilon$, $\nabla_\mu\vartheta_\nu \equiv \partial_\mu\vartheta_\nu - \Gamma^\lambda_{\mu\nu}\vartheta_\lambda$. Thus,

$$\begin{aligned}\widetilde{R}_{\mu\nu} = R_{\mu\nu} + (n-2)\nabla_\mu\vartheta_\nu + (\nabla\cdot\vartheta)\,g_{\mu\nu}\\ + (n-2)\left(\vartheta_\mu\vartheta_\nu - \vartheta^2 g_{\mu\nu}\right),\end{aligned} \tag{A4a}$$

$$\widetilde{\widehat{R}}_{\mu\nu} = \widehat{R}_{\mu\nu} + (n-2)\Big[\nabla_\mu\vartheta_\nu + \vartheta_\mu\vartheta_\nu - \frac{g_{\mu\nu}}{n}(\nabla\cdot\vartheta + \vartheta^2)\Big], \tag{A4b}$$

$$\widetilde{R} = \Omega^2\left[R + 2(n-1)\nabla\cdot\vartheta - (n-1)(n-2)\vartheta^2\right]. \tag{A4c}$$

Note that $\nabla\cdot\vartheta = \Box\Upsilon$. For $n = 4$, these become

$$\widetilde{R}_{\mu\nu} = R_{\mu\nu} + 2\nabla_\mu\vartheta_\nu + \big(\nabla\cdot\vartheta\big)g_{\mu\nu} + 2\big(\vartheta_\mu\vartheta_\nu - \vartheta^2 g_{\mu\nu}\big), \tag{A5a}$$

$$\widetilde{\widehat{R}}_{\mu\nu} = \widehat{R}_{\mu\nu} + 2\Big(\nabla_\mu\vartheta_\nu + \vartheta_\mu\vartheta_\nu\frac{g_{\mu\nu}}{4}\big(\nabla\cdot\vartheta + \vartheta^2\big)\Big), \tag{A5b}$$

$$\widetilde{R} = \Omega^2\left(R + 6\left(\nabla\cdot\vartheta - \vartheta^2\right)\right). \tag{A5c}$$

The Weyl tensor $C^\kappa{}_{\lambda\mu\nu}$ is invariant under conformal transformations, so that $\sqrt{g}C^2$ is also invariant. Assuming that the conformal transform does not change the topology (i.e., Euler characteristic), then G must change by $\nabla_\mu J^\mu$ for some current J^μ. Since $G = C^2 - 2W$, with $W \equiv \hat{R}^2_{\mu\nu} - R^2/12$, we find that W transforms as

$$\widetilde{W} - W = 4\nabla_\mu J^\mu, \quad \text{where} \tag{A6}$$

$$J_\mu \equiv \vartheta_\nu\nabla_\mu\vartheta^\nu - \vartheta_\mu\nabla\cdot\vartheta + \left(R_{\mu\nu} - g_{\mu\nu}\frac{R}{2}\right)\vartheta^\nu + \vartheta^2\vartheta_\mu. \tag{A7}$$

To derive this result, we must calculate

$$\sqrt{\widetilde{g}}\,\widetilde{W} = \sqrt{\widetilde{g}}\left(\widetilde{\widehat{R}}{}^2_{\mu\nu} - \frac{\widetilde{R}^2}{12}\right), \tag{A8a}$$

$$\begin{aligned}= \sqrt{g}\Bigg[\bigg[\widehat{R}_{\mu\nu} + 2\Big(\nabla_\mu\vartheta_\nu + \vartheta_\mu\vartheta_\nu -\\ \frac{g_{\mu\nu}}{4}\big(\nabla\cdot\vartheta + \vartheta^2\big)\Big)\bigg]^2 - \frac{\left[R + 6\left(\nabla\cdot\vartheta - \vartheta^2\right)\right]^2}{12}\Bigg].\end{aligned} \tag{A8b}$$

Hence, letting $\Delta W \equiv \widetilde{W} - W$, we find

$$\begin{aligned}\Delta W = {} & 4\widehat{R}^{\mu\nu}\left(\nabla_\mu\vartheta_\nu + \vartheta_\mu\vartheta_\nu\right) - R\left(\nabla\cdot\vartheta - \vartheta^2\right) + \\ & 4\left(\nabla_\mu\vartheta_\nu + \vartheta_\mu\vartheta_\nu - \frac{g_{\mu\nu}}{4}\left(\nabla\cdot\vartheta + \vartheta^2\right)\right)^2 - \\ & 3\left(\nabla\cdot\vartheta - \vartheta^2\right)^2, \end{aligned} \tag{A9a}$$

$$\begin{aligned} = {} & 4\widehat{R}^{\mu\nu}\nabla_\mu\vartheta_\nu - R\nabla\cdot\vartheta + 4\widehat{R}^{\mu\nu}\vartheta_\mu\vartheta_\nu + R\vartheta^2 + \\ & 4\left(\nabla_\mu\vartheta_\nu + \vartheta_\mu\vartheta_\nu\right)^2 - \left(\nabla\cdot\vartheta + \vartheta^2\right)^2 - \\ & 3\left(\nabla\cdot\vartheta - \vartheta^2\right)^2, \end{aligned} \tag{A9b}$$

$$\begin{aligned} = {} & \left(4R^{\mu\nu}\nabla_\mu\vartheta_\nu - 2R\,\nabla\cdot\vartheta\right) + \\ & 4\left(R^{\mu\nu}\vartheta_\mu\vartheta_\nu + \left(\nabla_\mu\vartheta_\nu\right)^2 - \left(\nabla\cdot\vartheta\right)^2\right) + \\ & 4\left(2\vartheta^\mu\vartheta^\nu\nabla_\mu\vartheta_\nu + \vartheta^2\nabla\cdot\vartheta\right). \end{aligned} \tag{A9c}$$

In the last step, the squares were expanded into monomials and the various terms gathered into a polynomial in ϑ_μ. For later convenience, the terms involving the curvature were expressed in terms of the usual Ricci tensor.

We now wish to show that the change ΔW can be written as the divergence of a vector. (Fortunately, the quartic terms involving $(\vartheta^2)^2$ canceled out in Eq. (A9c), as required.) The linear terms may be written as $4\nabla_\mu\left[(R^{\mu\nu} - g^{\mu\nu}R/2)\vartheta_\nu\right]$, since the Einstein tensor has zero divergence. The cubic terms are also easily seen to be $4\nabla_\mu\left[\vartheta^2\theta^\mu\right]$.

The quadratic terms require a bit more work. We will want to use the well-known relation[24] $R^{\mu\nu}\vartheta_\nu = [\nabla^\nu, \nabla^\mu]\vartheta_\nu = \nabla^\nu\nabla^\mu\vartheta_\nu - \nabla^\mu\nabla\cdot\vartheta$, in order to write it as gradients like the other terms. We may take advantage of the fact that ϑ_ν is itself the gradient of a scalar to rewrite $\nabla^\mu\vartheta_\nu = \nabla^\mu\nabla_\nu\Upsilon = \nabla_\nu\nabla^\mu\Upsilon = \nabla_\nu\vartheta^\mu$, since two covariant derivatives commute when acting on a scalar. Hence, $R^{\mu\nu}\vartheta_\nu = \Box\vartheta^\mu - \nabla^\mu\nabla\cdot\vartheta$.

To bring all the quadratic terms into the form of a divergence, note that there are only two vector monomials that can be formed that are both quadratic in ϑ_μ and have a single gradient, *viz.*, $\vartheta_\nu\nabla_\mu\vartheta^\nu$ and $\vartheta_\mu\nabla\cdot\vartheta$, so the quadratic terms in the current J^μ must be a linear combination of these two vectors. Their divergences are

$$\nabla_\mu\left(\vartheta_\nu\nabla^\mu\vartheta^\nu\right) = \left(\nabla_\mu\vartheta_\nu\right)^2 + \vartheta_\nu\Box\vartheta^\nu, \tag{A10a}$$

$$\nabla_\mu\left(\vartheta^\mu\nabla\cdot\vartheta\right) = \left(\nabla\cdot\vartheta\right)^2 + \vartheta^\mu\nabla_\mu(\nabla\cdot\vartheta). \tag{A10b}$$

Pulling all these pieces together, we find the quadratic terms become

$$\left[R^{\mu\nu}\vartheta_\mu\vartheta_\nu + \left(\nabla_\mu\vartheta_\nu\right)^2 - \left(\nabla\cdot\vartheta\right)^2\right] \tag{A11a}$$

$$\begin{aligned} = {} & \vartheta_\nu\Box\vartheta^\nu - \vartheta^\mu\nabla_\mu(\nabla\cdot\vartheta) + \nabla_\mu\left(\vartheta_\nu\nabla^\mu\vartheta^\nu\right) - \\ & \vartheta_\nu\Box\vartheta^\nu - \nabla_\mu\left(\vartheta^\mu\nabla\cdot\vartheta\right) + \vartheta^\mu\nabla_\mu(\nabla\cdot\vartheta) \end{aligned} \tag{A11b}$$

$$= \nabla_\mu\left[\vartheta_\nu\nabla^\mu\vartheta^\nu - \vartheta^\mu\,\nabla\cdot\vartheta\right], \tag{A11c}$$

establishing finally that ΔW is a total divergence. Thus, it contributes nothing to the EoM and, unlike the G-B term, also zero from the boundary of a compact manifold.

For example, in four dimensions, the Lagrangian density $\mathcal{L}_{ho}$, Eq. (4.2b), involving the real field $\phi(x)$ becomes

$$\begin{aligned}\mathcal{L}_{ho} = \sqrt{g}\Bigg(& \frac{1}{3b}\left[R - 6\,\Box\Upsilon + 6(\nabla\Upsilon)^2\right]^2 + \frac{1}{2a}C^2_{\kappa\lambda\mu\nu} + \\ & cR^*R^*\Bigg), \end{aligned} \tag{A12}$$

with $\Upsilon \equiv (1/2)\log(\phi^2/M^2)$. The last term, which takes the form of a divergence locally, can be ignored in perturbation theory. With the form of $\mathcal{L}_{ho}$ in Eq. (A12), the full action can then be written as

$$\begin{aligned} S_E = \int d^4x\sqrt{g}\Big[& -\frac{\xi M^2}{2}\big[R - 6\,\nabla^2\Upsilon + \\ & 6(\nabla\Upsilon)^2\big] + \mathcal{L}_{ho} + \mathcal{L}_J(\phi, g_{\mu\nu})\Big], \end{aligned} \tag{A13a}$$

$$\text{where } \mathcal{L}_J(\phi, g_{\mu\nu}) = \frac{ZM^2}{2}\left(\nabla\Upsilon\right)^2 + \frac{\lambda M^4}{4}. \tag{A13b}$$

The linear term in R now has Einstein-Hilbert form, and the original ϕ^4 self-interaction has become a cosmological constant that is positive for $\lambda > 0$! Having assumed that $\phi(x) \neq 0$, we may take $\phi > 0$, WLOG, since the action is invariant under $\phi \to -\phi$.

What remains is to gather like terms together in Eq. (A13). The terms quadratic in Υ are

$$\frac{1}{2}\left(Z + 6\xi\right)\left(M\nabla_\mu\Upsilon\right)^2, \tag{A14}$$

where we temporarily neglected other terms coming from $\mathcal{L}_{ho}$. Assuming that $Z + 6\xi > 0$, the canonically normalized scalar field is $\zeta \equiv \sqrt{(Z+6\xi)}\,M\Upsilon$. The preceding action then becomes Eq. (4.2) in the text.

We see that terms in $\mathcal{L}_{ho}$ involve powers of ζ/M. Similarly, if we carry out a derivative expansion in the metric as usual, then terms involving gradients of the metric beyond the quadratic terms and all those coming from $\mathcal{L}_{ho}$ will carry inverse powers of M. Thus, while $\mathcal{L}_{ho}$ is critical for renormalizability, the low energy effective theory at energy scales small compared to M will be dominated by the Einstein-Hilbert action as usual. Of course, M is completely arbitrary here, but eventually in the QFT, we hope to reconcile this with the observed value.

Appendix B: Lie algebra conventions

We briefly review our conventions [4] for the Lie algebra of $SO(10)$ in order to establish our notation and conventions. The defining or fundamental representation of the *group* $SO(10)$ consists of 10×10 real, orthogonal matrices, $\mathcal{O}$ satisfying $\mathcal{O}\mathcal{O}^\tau = 1$, where $\mathcal{O}^\tau$ denotes the transpose. Writing $\mathcal{O} = \exp(i\theta_a R^a)$, the (Hermitian)

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[24] More generally, $[\nabla_\mu, \nabla_\nu]V_\lambda = R^\kappa{}_{\lambda\mu\nu}V_\kappa$ which mathematicians [40] would write as $\nabla^2 V = RV$, the wedge product being understood. In this notation, $\nabla^2 \equiv \nabla\wedge\nabla \neq \Box$.

generators R^a must be imaginary and antisymmetric, satisfying

$$\left[R^a, R^b\right] = i f^{abc} R^c, \quad \mathrm{Tr}[R^a R^b] = \delta^{ab}/2, \tag{B1}$$

where we adopted the usual normalization convention (in physics) for the fundamental. Representation matrices are considered equivalent if they differ only by a unitary transformation $\widetilde{\mathcal{R}}^a = U^\dagger \mathcal{R}^a U$. This is because the transformed matrices $\widetilde{\mathcal{R}}^a$ are Hermitian and still satisfy Eq. (B1) with the same structure constants f^{abc}. On the other hand, these equivalent matrices may be neither real nor antisymmetric. In particular, it is possible to choose them so that the Cartan subalgebra is diagonal. (See, *e.g.,* Ref. [41].) This is of considerable advantage for analyzing the patterns of SSB.

In order to understand better the choice of basis described in Eq. (6.1) and thereafter, we can proceed as follows. The 10×10 Hermitian generators R^a are broken down into 4 5×5 blocks of the form given in Eq. (6.1). The $\mathcal{R}_1^a$ are 5×5 Hermitian matrices, of which there are 25 linearly independent possibilities. These 25 constitute a complete set that satisfy the algebra of $U(5){=}SU(5)\otimes U(1)$, with the Cartan subalgebra given by the diagonal generators. We associate them with the first 25 $SO(10)$ generators, defining $\mathcal{R}_1^a = 0$ for $\{a{=}26,\ldots,45\}$:

$$R_1^a \equiv \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \mathcal{R}_1^a & 0 \\ \hline 0 & -\mathcal{R}_1^{a*} \end{array}\right), \ \{a{=}1,\ldots,25\}. \tag{B2}$$

We choose the first 24 $\mathcal{R}_1^a$ to be traceless, generators of the $\mathbf{5}$ of $SU(5)$, with the 25th proportional to $\mathbf{1_5}$, the generator of the $U(1)$, normalized as required by $SO(10)$. This is frequently written as $\mathbf{5_{-2}}$. The 25 conjugate matrices $\{-\mathcal{R}_1^{a*}\}$ of $SU(5)\otimes U(1)$ are generators for $\overline{\mathbf{5}}_\mathbf{2}$.

On the other hand, we may also employ these generators to define the adjoint field,

$$\mathit{\Phi}_1 \equiv \sum_1^{25} \phi_a \mathcal{R}_1^a, \tag{B3}$$

with ϕ_a real. The components of the matrix $\mathit{\Phi}_1$ transform as the $\mathbf{24_0}\oplus\mathbf{1_0}$ representation of $SU(5)\otimes U(1)$.

The $\mathcal{R}_2^a$ are complex, antisymmetric, 5×5 matrices, of which there are 10 linearly independent that we shall call $\mathcal{R}^n$. Because these are antisymmetric, we have $R^{n\dagger} = -R^{n*}$. From these, we may form two sets of 10×10, Hermitian matrices,

$$R_2^{24+2n} \equiv \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} 0 & \mathcal{R}^n \\ \hline -\mathcal{R}^{n*} & 0 \end{array}\right), R_2^{25+2n} \equiv \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} 0 & i\mathcal{R}^n \\ \hline i\mathcal{R}^{n*} & 0 \end{array}\right), \tag{B4}$$

for $\{n{=}1,\ldots,10\}$. Although the sub-blocks are obviously not linearly independent, the two sets are linearly independent as $SO(10)$ generators. We define $R_2^a = 0$ for $\{a{=}1,\ldots,25\}$.

These too may be used to compose fields

$$\mathit{\Phi}_2 \equiv \sum_1^{10} (\phi_{24+2n} + i\phi_{25+2n}) \mathcal{R}^n, \tag{B5}$$

with real $(\phi_{24+2n}, \phi_{25+2n})$. In fact, it can be shown that $(\mathit{\Phi}_2)_{ij}$ transforms as the antisymmetric product representation $(\mathbf{5_{-2}}\otimes\mathbf{5_{-2}})_a = \overline{\mathbf{10}}_{\mathbf{-4}}$ of the $U(5)$. Consequently, $-(\mathit{\Phi}_2^*)_{ij} \equiv -(\mathit{\Phi}_2)^{ij}$ transforms as the conjugate representation $(\overline{\mathbf{5}}_\mathbf{2}\otimes\overline{\mathbf{5}}_\mathbf{2})_a = \mathbf{10_{+4}}$. (See, *e.g.,* Ref. [42], Tables 29 & 43.)

Combining these 45 component fields, we may write the adjoint of $SO(10)$ in the block form

$$\Phi = \sqrt{2}\,\phi_a R^a = \left(\begin{array}{c|c} \mathit{\Phi}_1 & \mathit{\Phi}_2 \\ \hline -\mathit{\Phi}_2^* & -\mathit{\Phi}_1^* \end{array}\right). \tag{B6}$$

Indeed, the preceding decomposition describes the branching rules for $SO(10)\to SU(5)\otimes U(1)$, *viz.,*

$$\mathbf{45} \to \mathbf{1_0} \oplus \mathbf{24_0} \oplus \overline{\mathbf{10}}_{\mathbf{-4}} \oplus \mathbf{10_{+4}}. \tag{B7}$$

The first two $\mathbf{1_0}, \mathbf{24_0}$ are self-conjugate, whereas the last two are distinct conjugate pairs. To break down this manner of representing $SO(10)$ into greater detail, since the $\mathcal{R}_1^a$ generate the algebra of $SU(5)\otimes U(1)$, which has rank five, we can choose generators of the Cartan subalgebra, $\mathcal{H}_1^i, i{=}\{1,\ldots,5\}$, to be diagonal. Setting $\mathcal{H}_2^i = 0$, the corresponding five $SO(10)$ generators are

$$H^i \equiv \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \mathcal{H}_1^i & 0 \\ \hline 0 & -\mathcal{H}_1^{i*} \end{array}\right), \ \{i{=}1,\ldots,5\}. \tag{B8}$$

With the appropriate normalization of the $U(1)$ generator, we may assume $\mathrm{Tr}[H^i H^j] = \mathrm{Tr}[\mathcal{H}_1^i \mathcal{H}_1^j] = \delta^{ij}/2$, as in Eq. (B1), *i.e.,* the H^i are the Cartan generators of $SO(10)$ as well. In the text, this was applied to the field $\mathit{\Sigma}$, decomposed as in Eq. (6.2). It immediately follows that the expectation values obey Eq. (6.5).

Similarly, the real vector fields, A^μ, which transform as the adjoint of $SO(10)$, may be defined analogously to Φ, Eq. (B6), $A^\mu \equiv \sqrt{2}\,A_a^\mu R^a$.

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